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Keywords: instrumental variables, local average treatment effects, event-study.

JEL Codes: C21, C23, C26.

THE INTERPRETATION OF 2SLS WITH A CONTINUOUS INSTRUMENT: A WEIGHTED LATE REPRESENTATION

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Abstract. This note introduces a novel weighted local average treatment effect representation for the two-stages least-squares (2SLS) estimand in the continuous instrument with binary treatment case. Under standard conditions, we obtain weights that are nonnegative, integrate to unity, and assign larger values to instrument support points that deviate from their average. Our representation does not require instruments to be discretized nor relies on limiting arguments, such as those used in the definition of the marginal treatment effect (MTE). The pattern of the weights also has a clear interpretation. We believe these features of the representation to be useful for applied researchers when communicating their results. As a direct byproduct of our approach, we also obtain a representation of the 2SLS estimand as a weighted average of treatment effects among "marginal compliance" groups, without having to resort to the threshold-crossing representation underlying the MTE construction. As an application, we consider the interpretation of "event-study 2SLS" specifications with continuous instruments.

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1. Introduction

Instrumental variable (IV) methods constitute one of the worhorses in the research toolkit of applied economists (Angrist and Krueger, 2001; Imbens, 2014; Abadie and Cattaneo, 2018). Since the seminal work of Imbens and Angrist (1994), henceforth IA, it is well known that, in a potential outcomes framework – and under exclusion, independence, relevance and monotonicity assumptions –, the instrumental variable estimator with binary treatment and binary instrument identifies a local average treatment effect (LATE) in the subpopulation whose treatment adoption is affected by the instrument (compliers). This representation has been further extended to accomodate discrete (Angrist and Imbens, 1995) and continuous (Angrist et al., 2000) treatments.

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Recent research on identification with instrumental variables methods in the potential outcomes framework has explored, *inter alia*, the nature of monotonicity assumptions under choice-theoretic perspectives (Heckman and Pinto, 2018; Mogstad et al., 2021); the interpetation of 2SLS estimands under different approaches to controlling for covariates (Kolesár et al., 2013; Blandhol et al., 2022; Słoczyński, 2022), and the interpretation of distinct weighting schemes underlying different IV estimands (Mogstad et al., 2018; Coussens and Spiess, 2021; Escanciano et al., 2023).

In this note, we revisit the interpretation of the 2SLS estimand in the case of a binary treatment and continuous instrument. In this setting, applied researchers are often faced with two distinct recommendations from the literature concerning the interpretation of 2SLS estimates. One branch advises researchers to *discretize* their instruments (Angrist and Pischke, 2009, p. 139-140), in which case available representations of IV estimands as weighted LATEs may be resorted to. Another branch recommends results be interpreted as weighted averages of marginal treatment effects (MTEs) (Heckman and Vytlacil, 2005; Heckman et al., 2006). These approaches have not been unanimously adopted, though. First, the use of continuous (undiscretized) instruments is quite common in the applied literature, as evidenced by the use of weather and distance-related instrumental variables. Second, even though the threshold-crossing model that motivates the definition of the MTE is known to be equivalent to the framework of IA Vytlacil (2002); and that, in the former, the MTE may be seen as a limit-form of the LATE; applied researchers are much more accostumed with non-limit forms of the LATE, often going large ways in interpreting their estimates as such, even in cases where such interpretation is not warranted (Blandhol et al., 2022).

This note aims to complement the literature by providing, to the best of our knowledge, a novel (non-limit) weighted LATE interpretation of 2SLS estimands in the continuous instrument *cum* binary treatment case. We provide conditions under which a class of Wald estimands that nests 2SLS as a particular case may be interpreted as a weighted-average of complier treatment effects, where compliance groups are defined with respect to treatment status at the (conditional on controls) average instrument value. The weights are nonnegative, integrate to unity, and assign larger values to instrument support points that deviate from the (conditional on controls) average. As we argue below, such patterns lead to weighted averages with an intuitive interpretation. Our results are derived under the same set of assumptions of IA (also Angrist et al., 1996), which we couple with the requirement that, in the specification adopted, covariates are controlled for in a sufficiently flexible manner. The latter restriction is known to be sufficient, under the IA assumptions, for 2SLS estimands to produce nonnegative weights (Blandhol et al., 2022). We further show that, as an immediate byproduct of our representation result, we obtain a representation of our class of Wald IV estimands in terms of "marginal compliance" groups, without having to resort to the the threshold-crossing model which generates the MTE representation.

As an application of our main result, we study the causal content of "event-study-IV" specifications with binary treatment and a continuous instrument. This type of specification, whose binary instrument counterpart is discussed in Hudson et al. (2017), leverages a time-invariant instrument to assess the effects of a time-varying policy. We provide conditions that ensure 2SLS estimands of this type of specification have a causal interpretation. We also provide conditions that justify the practice of assessing pre-trends in these settings.

The remainder of this note is organized as follows. Section 2 introduces the class of Wald estimands considered and presents the representation result. Section 3 applies the result in the context of event-study IV. Section 4 concludes.

2. Weighted LATE representation of a class of Wald Estimands under a continuous instrument

Suppose the researcher is interested in assessing the causal effect of a binary treatment $D \in \{0,1\}$ on an outcome Y. The researcher has access to a continuous scalar instrument Z, and a set of controls X. We assume that second moments of Y and Z exist. We consider a Wald estimand which aims to estimate the causal effect of D by leveraging covariation between Z and D, after partialling out covariation between Z and X. This estimand is given by:

$$\beta_{\text{Wald}} = \frac{\mathbb{E}[(Z-g^*(X))Y]}{\mathbb{E}[(Z-g^*(X))D]}\,, \tag{1}$$

where g^* is the $L_2(\mathbb{P})$ projection of Z on a space $\mathcal{G}$ of scalar functions of X with finite second moment, i.e.

$$g^* \in \operatorname{argmin}_{g\in\mathcal{G}} \mathbb{E}[(Z-g^*(X))^2]\,. \tag{2}$$

The Wald estimand (1) explores variation of Z after controlling for potential confounding due to X. These variables may be controlled for in a possibly nonlinear manner, for example by functions $g \in \mathcal{G}$ that vary nonlinearly in X. As a special case, if one considers a transformation p from X to $\mathbb{R}^k$, and takes $\mathcal{G} = \{\gamma \in \mathbb{R}^k : \gamma' p(X)\}$, then an application of the partitioned inverse formula shows that β_{Wald} is equal to the 2SLS estimand of β in the linear system:

$$\begin{aligned} D &= \alpha Z + \gamma' p(X) + u \\ Y &= \beta D + \omega' p(X) + v \end{aligned}\,. \tag{3}$$

We consider the interpretation of (1) in a potential outcomes framework. Following IA (also Angrist et al., 1996), we assume that:

Assumption 1. *We assume that:*

(1) ***Potential treatments:*** *observed treatment status is given by $D = D(Z)$, where $\mathcal{D} = \{D(z) : z \in \mathcal{Z}\}$ are the potential treatment statuses associated with different values of the instrument, with $\mathcal{Z} \subseteq \mathbb{R}$ denoting the instrument support.*

(2) ***Potential outcomes:*** *observed outcomes are given by* $Y = DY(1,Z) + (1-D)Y(0,Z)$*, where* $Y(d,z)$ *are the potential outcomes associated with treatment status* $d \in \{0,1\}$ *and instrument value* $z \in \mathcal{Z}$*.*

(3) ***Exclusion restriction:*** *for each* $d \in \{0,1\}$*,* $Y(d,z) = Y(d,z') =: Y(d)\ \forall z, z' \in \mathcal{Z}$*.*

(4) ***Conditional independence:*** *conditionally on* X*,* Z *is independent of* $\{Y(0), Y(1), \mathcal{D}\}$*.*

(5) ***Monotonicity:*** *either* $\mathbb{P}[D(z) \leq D(z')] = 1$ *for every* $z, z' \in \mathcal{Z}$ *with* $z \leq z'$*; or* $\mathbb{P}[D(z) \geq D(z')] = 1$ *for every* $z, z' \in \mathcal{Z}$ *with* $z \leq z'$*.*

The next proposition is our main result.

Proposition 1. *Suppose Assumption 1 holds. Let* $\psi(X)$ *be a version of* $\mathbb{E}[Z|X]$*. In addition, suppose the following conditions hold:*

(1) ***Support condition:*** *the support* $\mathcal{Z}$ *is given by* $[\underline{z}, \overline{z}]$*, where* $\underline{z}, \overline{z} \in \mathbb{R} \cup \{-\infty, \infty\}$*.*

(2) ***Moments:*** $Y(1)$*,*$Y(0)$ *and* Z *have finite second moments.*

(3) ***Relevance:*** $\mathbb{E}[(Z - g^*(X))D] \neq 0$*.*

(4) ***Flexible specification:*** $\psi \in \mathcal{G}$*.*

We then have that the estimand (1) is well-defined, and that:

$$\beta_{Wald} = \mathbb{E}[w(X,Z)\Delta(X,Z)]\,,$$

where

$$\Delta(x,z) = \mathbb{E}[Y(1) - Y(0)|X = x, D(z) \neq D(\psi(x))]\,,$$

and $w(x,z) = \frac{\omega(x,z)}{\mathbb{E}[\omega(X,Z)]}$*, with:*

$$\omega(x,z) = |z - \psi(x)|\mathbb{P}[D(z) \neq D(\psi(x))|X = x]\,.$$

Proof. See Online Appendix A. □

Proposition 1 shows that, under the stated assumptions, the Wald estimand (1) may be written as a weighted average of complier treatment effects. In the representation of Proposition 1, compliance is defined with respect to potential treatment status at the average instrument value $\mathbb{E}[Z|X = x]$, with weights being attached to average treatment effects in subpopulations that would change their treatment status upon being offered a shift of instrument value from $\mathbb{E}[Z|X = x]$ to z, for different values of $z \in [\underline{z}, \overline{z}]$. The weights in the representation are nonnegative and average to unity. Moreover, due to the monotonicity assumption, the weights $w(x,z)$ are nonincreasing in z for $z < \mathbb{E}[Z|X = x]$ and nondecreasing in z for $z > \mathbb{E}[Z|X = x]$. Consequently, the weighted average assigns larger weights to more extreme values of z.

The representation in Proposition 1 hinges crucially on the interval support assumption on the instrument, as it ensures that the potential treatment status at the average $\mathbb{E}[Z|X = x]$ is well-defined. In addition, we note that, since the representation uses a (conditional on X) fixed reference potential treatment status in defining compliers, it may be easier to interpret than varying-reference-level ones (Cornelissen et al., 2016). To see this point, first note that our fixed-level representation involves overlapping subpopulations. Indeed, for any $z' > z > \psi(x)$, $\Delta(x, z')$ includes in its average all compliers averaged in $\Delta(x, z)$; the same holding true for $z' < z < \psi(x)$. Since more extreme values of the instrument are precisely those assigned larger weights in the representation, it follows that the representation assigns larger weights to the most encompassing complier subpopulations, a useful feature in the interpretation of results.

In spite of the attractiveness of our representation, it should be noted that, in some settings, it may be also useful to interpret the estimand as a weighted average of treatment effects in disjoint subpopulations. As we show below, one immediate corollary of Proposition 1 is a representation of (1) in terms of nonoverlapping "marginal compliance" groups. Interestingly, our result follows without having to resort to the threshold-crossing representation that motivates the MTE, which may be a further useful feature for applied researchers in communicating their empirical results.

In what follows, define the marginal compliance group of an individual as the variable C that equals c if:

$$\lim_{z \uparrow c} D(z) \neq D(c) \,,$$

i.e. C is the smallest instrument value required for an individual to change her behavior. If the individual is not a complier, we set $C = \varnothing$.

Corollary 1. *Suppose that the Assumptions required in Proposition 1 hold. Suppose that $C|X, C \neq \varnothing$ admits a regular conditional Lebesgue density $f^*_{C|X}(\cdot|\cdot)$. We then have that:*

$$\beta_{Wald} = \int \int_{\underline{z}}^{\overline{z}} w^*(x, c)\Delta^*(x, c) \; dc \; \mathbb{P}_X(dx) \,,$$

where

$$\Delta^*(x, c) = \mathbb{E}[Y(1) - Y(0)|X = x, C = c] \,,$$

and $w^*(x, c) = \frac{\omega^*(x,c)}{\int \int_{\underline{z}}^{\overline{z}} \omega^*(a,b) \; db \; \mathbb{P}_X(da)}$, *with*

$$\omega^*(x, c) = \begin{cases} \mathbb{E}[|Z - \psi(x)|\mathbf{1}\{Z \leq c\}|X = x]f^*_C(c|x), & \text{if } c < \psi(x) \\ \mathbb{E}[|Z - \psi(x)|\mathbf{1}\{Z \;\; \geq c\}|X = x]f^*_C(c|x), & \text{if } c \geq \psi(x) \end{cases}$$

Proof. Note that

$$\mathbb{E}[(Y(1) - Y(0))\mathbf{1}\{D(z) \neq D(\psi(x))\}|X = x] = \int_{\underline{z}}^{\overline{z}} \Delta^*(x, c)\mathbf{1}\{c \in [z, \psi(x)] \cup [\psi(x), z]\}f^*_C(c|x)dc \,.$$

The conclusion then follows from Fubini theorem. □

Remark 1 (On the role of the flexible specification assumption)**.** Proposition 1 is derived under the requirement that the function class $\mathcal{G}$ is flexible enough so as to contain the conditional expectation function $\mathbb{E}[Z|X = x]$. It has been recently shown that this assumption is sufficient, in the IA setup, for 2SLS estimands to be represented as weighted averages of complier treatment effects with nonnegative weights (Blandhol et al., 2022). In the case that X has a finite number of support points, the assumption is satisfied by relying on a saturated $\mathcal{G}$, i.e. by considering $p(X)$ as a vector of indicator functions of all possible values x in the support of X. In settings where X is more complex, e.g. it contains continuous entries or a finite but very large number of support points, it may be preferable to rely on machine-learning methods that estimate representation (1) while flexibly controlling for X (Belloni et al., 2012; Chernozhukov et al., 2018).

3. Application: Event-Study IV with continuous instrument

In this, section, we explore a setting where a researcher has access to a panel spanning periods $t \in \{-T_0, -(T_0+1), \ldots, 0, 1, \ldots, T_1\}$. The researcher would like to assess the causal effect of a time-varying binary policy on an outcome Y_{it} of interest. She has access to a continuous, *time-invariant* instrument Z_i, as well as a set of time-invariant controls X_i. Consider the case where treatment starts at period $t = 1$ for a subset of the units in the population, and identify this group by the indicator $D_i = 1$. In these settings, a common approach consists in considering the following linear system:

$$Y_{it} = \alpha_i + \gamma_t + \sum_{\tau \neq 0} \beta_\tau \mathbf{1}\{t = \tau\} D_i + \sum_{\tau \neq 0} \mathbf{1}\{t = \tau\} \gamma_\tau' X_i + \epsilon_{i,t}\,, \tag{4}$$

which is estimated by 2SLS, instrumenting $\{\mathbf{1}\{t = \tau\}D_i\}_{\tau \neq 0}$ with $\{\mathbf{1}\{t = \tau\}Z_i\}_{\tau \neq 0}$. This approach may be interpreted as an event-study IV design, where the full-path of pre-treatment $\{\beta_\tau : \tau < 0\}$ and post-treatment effects $\{\beta_\tau : \tau > 0\}$ is identified by generating a set of instruments through the interaction of the time-invariant Z_i with time dummies. The specification controls for unit and time effects, as well as for differential trends according to the value of X_i.[1]

We consider the causal interpretation of the β_τ in the aforementioned approach in a potential outcomes framework. For that, we note that, for each $\tau \neq 0$, the 2SLS estimand β_τ in (4) is equal to the 2SLS estimand β_τ in the following system:

$$\begin{aligned} D_i &= c + \kappa Z_i + \pi' X_i + u_{i,t} \\ Y_{i,\tau} - Y_{i,0} &= \omega_\tau + \beta_\tau D_i + \theta_\tau' X_i + v_{i,t,\tau} \end{aligned} . \tag{5}$$

[1]We do not consider the inclusion of time-varying controls X_{it} in the specification, as such inclusion may generate negative weights in the representation of the estimand as an average treatment effect even in simple settings (Caetano and Callaway, 2023).

This shows that the event-study IV design is a version of the instrumented differences-in-differences approach discussed in Hudson et al. (2017).

The following proposition gives a causal interpretation for the estimands β_τ.

Proposition 2. *Suppose the following conditions hold:*

(1) ***potential treatments and instrument support:*** *observed treatment status is given by* $D_i = D_i(Z)$, *where* $\mathcal{D}_i = \{D_i(z) : z \in [\underline{z}, \overline{z}]\}$ *are the potential treaments statuses, with* $[\underline{z}, \overline{z}]$, $\underline{z}, \overline{z} \in \mathbb{R} \cup \{-\infty, \infty\}$, *denoting the instrument support.*

(2) ***potential outcomes and exclusion restriction:*** *observed outcomes are given by:* $Y_{it} = Y_{it}(0)$ *if* $t \leq 0$ *and* $Y_{it} = D_i Y_{it}(1) + (1 - D_i) Y_{it}(0)$ *if* $t > 0$, *where* $Y_{it}(d)$, $d \in \{0, 1\}$, *are the potential outcomes associated with the binary policy.*

(3) ***(conditional) independence from potential treaments:*** $\mathbb{P}[D(z) = 1 | X, Z] = \mathbb{P}[D(z) = 1 | X]$ *for every* $z \in [\underline{z}, \overline{z}]$.

(4) ***monotonicity:*** *either* $\mathbb{P}[D(z) \leq D(z')] = 1$ *for every* z, z' *with* $z \leq z'$; *or* $\mathbb{P}[D(z) \geq D(z')] = 1$ *for every* z, z' *with* $z \leq z'$.

(5) ***moments:*** *all random variables admit finite second moment.*

(6) ***first-stage:*** *in* (5), $\pi \neq 0$.

(7) ***linearity:*** $\mathbb{E}[Z_i | X_i] = a + b' X_i =: \psi(X_i)$.

We then have that:

- *for* $\tau < 0$, *if the following additional condition holds:*
  - ***absence of pre-trends:*** $\mathbb{E}[Y_{i,\tau}(0) - Y_{i,0}(0) | \mathcal{D}_i, X_i, Z_i] = \mathbb{E}[Y_{i,\tau}(0) - Y_{i,0}(0) | \mathcal{D}_i, X_i]$,

  then

$$\beta_\tau = 0\,.$$

.

- *for* $\tau > 0$, *if the following additional condition holds:*
  - ***parallel trends:*** $\mathbb{E}[Y_{i,\tau}(0) - Y_{i,0}(0) | \mathcal{D}_i, X_i, Z_i] = \mathbb{E}[Y_{i,\tau}(0) - Y_{i,0}(0) | \mathcal{D}_i, X_i]$ *and* $\mathbb{E}[Y_{i,\tau}(1) - Y_{i,0}(0) | \mathcal{D}_i, X_i, Z_i] = \mathbb{E}[Y_{i,\tau}(1) - Y_{i,0}(0) | \mathcal{D}_i, X_i]$

  then

$$\beta_\tau = \mathbb{E}[w_\tau(X_i, Z_i) \Delta_\tau(X_i, Z_i)]\,,$$

  where

$$\Delta_\tau(x, z) = \mathbb{E}[Y_{i,\tau}(1) - Y_{i,\tau}(0) | X_i = x, D_i(z) \neq D_i(\psi(x))]\,,$$

  and $w_\tau(x, z) = \frac{\omega_\tau(x,z)}{\mathbb{E}[\omega_\tau(X_i, Z_i)]}$, *with:*

$$\omega_\tau(x, z) = |z - \psi(x)| \mathbb{P}[D_i(z) \neq D_i(\psi(x)) | X_i = x]\,.$$

Proof. The proof of the proposition follows the same steps of the proof of Proposition 1, by working with outcomes in first-differences and observing that the mean independence assumptions are sufficient for establishing the weighted average representation. □

Proposition 2 provides a weighted LATE representation for the event-study IV specification. Our results clarify the type of parallel trends assumption that justifies a causal interpretation of the estimand. They also provide a justification for applied researchers to assess pre-trends in these settings.

4. Conclusion

This note introduces a weighted LATE representation for the 2SLS estimand in the binary treatment with continuous instrument case. We have shown our non-limit representation has an intuitive interpretation. In Section 3, we apply our representation in the context of event-study IV designs with continuous instruments, thus providing sufficient conditions for estimands to have a causal interpretation.

Data availability

No data was used for the research described in the article.

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Online Appendix of "The interpretation of 2SLS with a continuous instrument: a weighted LATE representation"

Appendix A. Proof of Proposition 1

Without loss of generality, we consider the monotonicity assumption in the direction $\mathbb{P}[D(z) \leq D(z')] = 1$ for every $z \leq z'$. Otherwise, we can redefine the instrument as $Z^* = -Z$ and the following proof would also apply.

First, we note that, by the moment and relevance assumptions, the numerator and denominator of β_{Wald} are well-defined. Next, we observe that the numerator is given by:

$$\begin{aligned}
&\mathbb{E}[(Z - g^*(X))Y] = \\
&\mathbb{E}[(Z - g^*(X))Y(0)] + \mathbb{E}[(Z - g^*(X))D(Y(1) - Y(0))] = \\
&\mathbb{E}[(Z - g^*(X))\mathbb{E}[Y(0)|X]] + \mathbb{E}[(Z - g^*(X))D(Y(1) - Y(0))] \,,
\end{aligned}$$

where the last equality uses iterated expectations, followed by the independence assumption. Now, it follows from the flexible specification assumption that $g^* = \psi$, from which we have that $\mathbb{E}[(Z - g^*(X))\mathbb{E}[Y(0)|X]] = 0$.

As for the second term, we note that, for support points (x, z):

$$\begin{aligned}
&\mathbb{E}[(Z - g^*(X))D(Y(1) - Y(0))|Z = z, X = x] = \\
&(z - \psi(x))\mathbb{E}[D(z)(Y(1) - Y(0))|X = x] = \\
&(z - \psi(x))\mathbb{E}[(D(z) - D(\psi(x)))(Y(1) - Y(0))|X = x] + \\
&(z - \psi(x))\mathbb{E}[D(\psi(x))(Y(1) - Y(0))|X = x] \,,
\end{aligned} \tag{6}$$

where the last equality added and subtracted $D(\psi(x))$ inside the expectation. Observe that, for $\mathbb{P}_X$-almost every x, $D(\psi(x))$ is well-defined, since $\mathbb{P}[\psi(X) \in [\underline{z}, \overline{z}]] = 1$. Now, suppose $z > \psi(x)$. In this case, by the monotonicity assumption, $D(z) - D(\psi(x))$ takes either value 0 (always- or never-taker in the comparison between instrument values $\psi(x)$ and z), or 1 (complier). Consequently, in this case:

$$\begin{aligned}
&(z - \psi(x))\mathbb{E}[(D(z) - D(\psi(x)))(Y(1) - Y(0))|X = x] = \\
&\underbrace{(z - \psi(x))}_{=|z=\psi(x)|}\mathbb{P}[D(z) \neq D(\psi(x))|X = x]\mathbb{E}[Y(1) - Y(0)|X = x, D(z) \neq D(\psi(x))] \,.
\end{aligned}$$

Symmetrically, if $z < \psi(x)$, $D(z) - D(\psi(x))$ takes either value 0 or -1, with the latter corresponding to compliance at the comparison between instrument values $\psi(x)$ and z. Consequently, we have that, in this case:

$$(z-\psi(x))\mathbb{E}[(D(z)-D(\psi(x)))(Y(1)-Y(0))|X=x]=$$
$$\underbrace{-(z-\psi(x))}_{=|z-\psi(x)|}\mathbb{P}[D(z)\neq D(\psi(x))|X=x]\mathbb{E}[Y(1)-Y(0)|X=x,D(z)\neq D(\psi(x))]\,.$$

Combining the above results, and using that, for $\xi(x) \coloneqq \mathbb{E}[D(\psi(x))(Y(1)-Y(0))|X=x]$, $\mathbb{E}[(Z-\psi(X))\xi(X)]=0$, we have that:

$$\mathbb{E}[(Z-g^*(X))Y]=\mathbb{E}[\omega(X,Z)\Delta(X,Z)]\,.$$

A similar argument then shows that the denominator of β_{Wald} is equal to $\mathbb{E}[\omega(X,Z)]$, which proves the desired result.