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An analysis of the distributive effects of public policies and their spillovers

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An analysis of the distributive effects of public policies and their spillovers *
(Preliminary Version)

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Abstract

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1 Introduction

In several areas of knowledge the main research question is to estimate the causal effect of a treatment variable (D) on an outcome variable (Y). Using the potential outcome model (Rubin 1974), define the potential outcome for the individual i to be treated or untreated (control) respectively as $Y_i(1)$ and $Y_i(0)$. The individual impact and the Average Treatment Effects (ATE) can be denoted respectively as $Y_i(1) - Y_i(0)$ and $ATE = \mathbb{E}[Y_i(1) - Y_i(0)]$. Unfortunately it is impossible to observe the same individual in both situations (treated and control), that is, we have only $Y = D_iY_i(1) + (1 - D_i)Y_i(0)$ where $D_i \in \{0, 1\}$. Thus, under the following assumption $(Y_i(1), Y_i(0)) \perp D_i$, we can use the sample analogues to estimate the average causal impact: $ATE = \mathbb{E}[Y|T = 1] - \mathbb{E}[Y|T = 0]$.

For the identification of the object ATE the assumption $(Y_i(1), Y_i(0)) \perp D_i$ has been used. In general this assumption will be satisfied when individuals are randomized to receive the treatment. When we performing a Randomized Control Trial (RCT) we can obtain a counterfactual that representing the treatment group: the randomization mechanism provides the balancing of the characteristics of the individuals excluding the possibility of selection in observable or not observable in both groups.

The causal effect of the treatment can be heterogeneous over the distribution of the outcome variable and distributive measures may become interesting to the researcher or policy maker. In this case, the Quantile Treatment Effect (QTE) estimators can capture different effects on the tails or in the middle of the distribution of Y. When we have an RCT we can directly use the measures proposed by Doksum (1974) and Lehmann (1975) defined as, for a given percentile $\tau \in (0, 1)$, the horizontal distance between the cumulative distribution functions of treatment (F_1) and control (F_0) groups: $QTE(\tau) = F_1^{-1}(\tau) - F_0^{-1}(\tau)$. Thus we have a complete description of the treatment effect for the entire distribution of the outcome variable.

The identification assumptions of ATE and QTE assume that the potential outcome of the individual i is not affected by the treatment selection mechanism and that the treatment received by an individual does not change the potential outcome of another individual excluding interaction between units and possible effects of spillover (Cox 1958, Imbens & Wooldridge 2009, Rubin 1980). This assumption, known as Stable Unit Treatment Value Assumption (SUTVA), may not be satisfied when, for example, treatment selection occurs at an individual level but an RCT occurs at an aggregate level (with intersection among groups) allowing the transmission of information or contagion between the units of the treatment or control groups [1]. In this case, when the the SUTVA is not

[1] One possible example is the Mexican income transfer program: Progresa. The RCT was carried out at the aggregate level of villages in poor rural areas of Mexico. As the villages are not isolated the intersection between the groups, in this case areas, allowed interaction between the individuals generating program spillover for several control groups.

valid, the estimators of interest will be biased(Sobel 2006).

Thus it is necessary to rewrite the SUTVA assumption to allow interaction among individuals at some level, such as interference within groups (households, schools, firms, etc.), and excluding contacts among them. In this case it is possible to recover the desirable properties of the objects of interest and also estimate other effects.

The seminal work of Halloran & Struchiner (1991) defines the concepts currently used to estimate spillover: direct, indirect, total and overall effects. The authors argue that depending on how the treatment and control groups are formed and, due to the interactions between individuals, the direct effects can be calculated incorrectly. In this case it is important to measure the indirect effects to account for how much public policy has spillover.

Using the random voucher experiment of the Move to Oportunity (MTO) program, Sobel (2006), it is shown that it is not possible to estimate the measures defined in Halloran & Struchiner (1991) without using additional assumptions, otherwise the results will be biased due to spillovers and intersections between the groups. In this case it is necessary to partition the clusters and allow an individual to affect the outcome variable of another individual only if they are within the same group naming this assumption as "partial interference". Given this assumption, and assuming that there is a two level random experiment, the author demonstrates the non bias of several spillover estimators.

From the works of Halloran & Struchiner (1991) and Sobel (2006) the literature was divided into works that contributed to demonstrate the properties of small and large samples. In the case of properties of small samples, the authors assume that there is a randomization design in two levels: at first stage the proportion or quantity of treated individuals is randomized into several groups, then in the second stage, the subjects are randomly selected within the group.

In this context, Hudgens & Halloran (2008) assume some assumptions, among them partial interference, and formalize from the potential outcomes, the direct, indirect, total and overall effects proving the absence of bias for all estimators. Tchetgen Tchetgen & VanderWeele (2010) use the same previous assumptions and extend the work of Hudgens & Halloran (2008) by calculating confidence intervals and defining the inverse probability weighting estimator, finally, Basse & Feller (2017) does no use the assumption of groups with homogeneous sizes and propose unbiased estimators for average effects and variance.

Recently other authors have proved the asymptotic properties for the spillover effects. Unlike previous authors Vazquez-Bare (2017) does not assume the existence of two stage randomization to identify the parameters of interest. He estimates the objects non-parametrically, performs inference, and demonstrates the asymptotic properties. Lastly, Leung (2017) does not assume SUTVA assumption treating the problem as a single network allowing interference between groups (cross-cluster links). The author establishes the conditions for the estimators to be consistent and asymptotically normal.

The above works has concentrated efforts to correctly measure spillover effects from the definitions of Halloran & Struchiner (1991). Philipson (2000) defines the seminal concepts of the external and private average effects of treatment due to variations in the proportion of treated. To identify the populational parameters, the author states that it is necessary to conduct a randomized two level experiment to generate random variation between groups and between individuals. According to the author the main difference with respect to Halloran & Struchiner (1991) is the incorporation of the change in distribution, for the average, due to variation of the proportion of treated.

Despite the advances made, there are still no papers formalizing the distributive effects of treatment spillover in two stage randomized experiments. Thus, in this context, the following general questions arise: i) how does to correctly define and identify the direct and indirect effects of treatment for quantiles? ii) what is the effect of the proportion of treated over the distribution of the outcome variable?

This work is interested in the saturation of the treatment by groups (proportion of treated) and uses the assumptions related to multilevel experiments to identify the effects of spillover (the details of the saturation randomization design will be discussed in the next chapter). The first central idea is to define, identify and estimate the direct and indirect effects, similiar to the average effects of the Halloran & Struchiner (1991), of treatment on the percentiles of the outcome variable. The effects of the interaction between the individuals may be different along the distribution, because depending on the type of program, the left tail, center or right tail may be affected in an unequal way. Thus, it becomes essential to have quantile estimators that allow to correctly identify these distributive effects.

In addition to identifying these objects, the private and spillover effects for quantiles, similar to the average effects of the Philipson (2000), are also define and identified from the proportion of treated. In general, the budget constraint of the policy maker does not allow the treatment of 100% of the target public. Thus, works related to the effects of the proportion on quantiles can help in the elaboration of public policies. It may not be necessary to treat the entire target public, since the interaction between individuals may overflow the results for the other members and this effect may be different throughout the distribution.

The main contributions of this work are: i) define and identify the direct and indirect effects of the proportion of treated individuals for quantiles; ii) separate the variation in the quantiles, due to the proportion of treated, into two parts called external and private effects, identifying this objects.

In addition to this introduction, this paper presents the potential outcomes and their assumptions in section 2, the objects of interest and their identifications are presented in section 3, and section 4 addresses the final considerations.

2 Potential outcomes and assumptions

The aim of this chapter is to define the potential outcomes and the assumptions in the context of *Random Saturation Design* (RSD).

RSD allocate the individuals in treated and control groups when there are two or more level of allocation, the so-called saturation level (Baird, Bohren, Mcintosh & Ozler 2015, Hahn, Hirano & Karlan 2011, Hirano & Hahn 2009, Hudgens & Halloran 2008, Sinclair 2012, Sinclair, McConnell & Green 2012). Let i, g and n represent respectively the individuals, $i = 1, 2, ..., n_g$, groups of allocation, $g = 1, 2, ..., G$, and the total number of individuals $n = n_1 + n_2 + ... + n_G$. Denote $\pi \in \Pi \subset [0, 1]$ the proportion of treated individuals, saturation level (S), where Π represent the support of saturation. The saturation level can be measured in the whole experiment, $S = \pi_o$, or inside each group g, $S = \pi_g$.

In the first stage of RSD the proportion of treated individuals, π_g, are allocated randomly across several different groups. Because in most setup, G is relatively small, one typically chooses few different values of π_g. It is important to notice that those G groups are not randomly formed and existed before the intervention.

In the second stage, for each group g, one selects randomly $\pi_g n_g$ subjects to form the "treatment group" of g. The remainder $(1 - \pi_g)n_g$ are not treated, and form the "control group" of g.

Define the variable C_{ig} as the one that assigns the value of the group (cluster) the individual belongs to. In other words, $C_{ig} = g$ for all individuals in cluster g. The treatment assignment will be denoted as D_{ig} and may assume values $d \in \{0, 1\}$. The probability of being treated among all groups is $\pi_o = \mathbb{P}(D_{ig} = 1)$. The probability of being treated given that $C_{ig} = g$ is $\pi_g = \mathbb{P}(D_{ig} = 1|C_{ig} = g)$.

Following the notation similar to Vazquez-Bare (2017) denote as $\mathbf{D}_g = [D_{1g}, D_{2g}, ..., D_{n_g g}]'$ the vector of length n_g of treatment assignments. The vector of size $n_g - 1$ containing the treatment status of the neighbors of i is given by $\mathbf{D}_{(i)g} = [D_{1g}, D_{2g}, ..., D_{i-1g}, D_{i+1g}, ..., D_{n_g-1g}]'$, so $\mathbf{D}_g = [D_{ig}, \mathbf{D}_{(i)g}]'$. The realization of vector $\mathbf{D}_{(i)g}$ is denoted as $\mathbf{d}_{(i)g} = [d_{1g}, d_{2g}, ..., d_{i-1g}, d_{i+1g}, ..., d_{n_g-1}g]' \in \mathcal{D}_g \subseteq \{0, 1\}^{n_g-1}$ and we have therefore $\mathbf{d}_g = [d, \mathbf{d}_{(i)g}] = [d, d_{1g}, d_{2g}, d_{3g}, ..., d_{i-1g}, d_{i+1g}, ..., d_{n_g-1g}]'$. The vector of all treatment assignments in all groups is $\mathbf{D} = [\mathbf{D}_1, \mathbf{D}_2, ..., \mathbf{D}_G]'$.

Let Y_{ig} be the outcome variable for individual i at group g. It can be written as:

$$Y_{ig} = Y_{ig}(\mathbf{D}) = Y_{ig}(\mathbf{D}_1, \mathbf{D}_2, ..., \mathbf{D}_{g-1}, \mathbf{D}_g, \mathbf{D}_{g+1}, ..., \mathbf{D}_{G-1}, \mathbf{D}_G) \tag{1}$$

Equation (1) depends on the entire vector of treatment assignments. We then impose some assumptions in order to precisely write Y_{ig} as a function of potential outcomes. Then the first restriction in equation (1) is given by:

Assumption 1 (A1: SUTVA at group level)**.** *There is interference only within the group*

$$Y_{ig}(\mathbf{D}_1, \mathbf{D}_2, ..., \mathbf{D}_G) = Y^*_{ig}(\mathbf{D}_g)$$

This assumption is the SUTVA extension that excludes spillover between groups, but allows spillover within the group. In this case, the outcome variable remains unchanged for all possible values of treatment outside the group the individual belongs to. Whether these groups have some contact, such as in neighborhoods or households, this assumption may not be valid, but if the groups are disjoint, such as schools, then there is probably no spillover between this units. Several authors have also used this approach, and each one adapt the SUTVA assumption conform their question (Baird, Bohren, Mcintosh & Ozler 2015, Graham, Imbens & Ridder 2010, Hudgens & Halloran 2008, Liu & Hudgens 2014, Manski 2013, Sobel 2006). For example, Manski (2013, p.4-5) and Sobel (2006, p.1405), call this assumptions respectively as "*constant treatment response*" and "*partial interference assumption*". For a realization $\mathbf{D}_g = [d, \mathbf{d}_{(i)g}] = [d, d_{1g}, d_{2g}, d_{3g}, ..., d_{i-1g}, d_{i+1g}, ..., d_{n_g-1g}]'$ and under the assumption **A1** the equation (1) can be rewritten as:

$$Y_{ig}(\mathbf{D}_g) = Y^*_{ig}(d, \mathbf{d}_{(i)g}) \tag{2}$$

This equation depend on the vector $\mathbf{d}_{(i)g}$ that contains information about the treatment assignment for all individuals within the same group. Individual i is connected to all other $n_g - 1$ individuals in that group by a given network structure. Thus, the fact that a given individual j and not individual k is treated affects i differently than another configuration in which individual k and not individual j is treated, where both j and k belong to the same group g as i. In other words the identity of the individual, who is being treated, matters.

In this context, if one does not have any information on the network structure, it is necessary to discuss what kind of assumptions about the potential outcomes are necessary to avoid using that information.

We can rewriting this structure for include the treatment saturation and reduce the dimensionality of equation (2). Thus define the potential outcomes for individual i at the saturation level $S = \pi_g$ for treated and control status respectively as:

$$\begin{gathered} Y^{**}_{ig}(1, S = \pi_g) \\ Y^{**}_{ig}(0, S = \pi_g) \end{gathered} \tag{3}$$

then consider the following assumption:

Assumption 2 (A2: interference between individuals inside the group)**.** *For a realization*

$\mathbf{D}_g = [d, \mathbf{d}_{(i)g}]$ *there is a function* $Y_{ig} : \{0,1\} \times \{0,1\}^{n_g - 1} \to \mathbb{R}$ *such that*

$$\mathbb{E}[Y^*_{ig}(d, \mathbf{d}_{(i)g})|S = \pi_g] = \mathbb{E}[Y^{**}_{ig}(d, \pi_g)|S = \pi_g]$$

The **A2** hypothesis says that, conditional in saturation level, the mean potential outcome for individual i at group with treatment vector $\mathbf{d}_{(i)g}$ will be equal to the mean potential outcome with saturation level at $S = \pi_g$.

Note that perhaps a previous network may exist, but the randomly experiment become information coming from the contacts irrelevant to explain the observed outcome, additionally, it is also implicit that the treatment does not induce the formation of new networks. Thus, the two level experiment and the above assumption allows to ignore the lack of information about network structure and, in this case, only the proportion of treated units (saturation) become relevant for determine the potential outcomes.

Consider the following example related to **A2** assumption: suppose there is a group with three individuals, with saturation level at $S = 2/3$ and that individual i has been treated. Then, the potential outcome can be write as $Y^*_{ig}(d, d_1, d_2)$. Thus, assume **A2** is equivalent to stating that $\mathbb{E}[Y^*_{ig}(1,1,0)|\pi_g = 2/3] = \mathbb{E}[Y^*_{ig}(1,0,1)|\pi_g = 2/3] = \mathbb{E}[Y^{**}_{ig}(1, \pi_g)|\pi_g = 2/3]$. Whether, for example, we reduce the saturation level for the level $S = 1/3$ and suppose that individual i was not treated, then $\mathbb{E}[Y^*_{ig}(0,1,0)|\pi_g = 1/3] = \mathbb{E}[Y^*_{ig}(0,0,1)|\pi_g = 1/3] = \mathbb{E}[Y^{**}_{ig}(0, \pi_g)|\pi_g = 1/3]$

The **A2** assumption reduces the dimensionality of the potential outcome and the codomain of $Y^*_{i,g}$ become $Y^{**}_{ig} : [0,1] \times \{0,1\} \to \mathbb{R}$. Thus, assuming **A2** we can rewrite the equation (2) as:

$$Y_{ig} = Y^{**}_{ig}(d, \pi_g) \tag{4}$$

the equation (4) shows that individual's potential outcome will depend on his own treatment status, but will also be a function of the a proportion individuals treated within his your group. Using this structure we would like to compare, for example, the potential outcome of individual i when he participates in the treatment with his respective counterfactual: $Y^{**}_{ig}(1, S = \pi_g) - Y^{**}_{ig}(0, S = \pi_g)$.

Under **A1** and **A2** assumption the observed outcome of the individual i within group g if $S = \pi_g$ can be written as:

$$Y_{ig} = Y^{**}_{ig}(1, S = \pi_g)D_{ig} + Y^{**}_{ig}(0, S = \pi_g)(1 - D_{ig}) \tag{5}$$

additionally we can rewrite equation (5) for the individual i as:

$$Y_i = \sum_{g=1}^{G} [Y^{**}_{ig}(1, S = \pi_g)D_{ig} + Y^{**}_{ig}(0, S = \pi_g)(1 - D_{ig})]\mathbb{I}\{C_{ig} = g\} \tag{6}$$

where $\mathbb{I}\{\}$ is the indicator function which assumes the value 1 if the individual i is observed in the group g.

Hence, if $Y_{ig}^{**}(d, S = \pi_g)$ is independent of D_{ig} given $S = \pi_g$, and $d = 0, 1$, then

$$\mathbb{E}[Y_{ig}^{**}(d, S = \pi_g)|D_{ig} = d, S = \pi_g] = \mathbb{E}[Y_{ig}^{**}(d, S = \pi_g)|S = \pi_g]$$

This is ensured if there is randomization of who will receive the treatment, since the proportion is fixed as $S = \pi_g$. Finally, because of the randomization of the proportion π_g among the groups

$$\mathbb{E}[Y_{ig}^{**}(d, S = \pi_g)|S = \pi_g] = \mathbb{E}[Y_{ig}^{**}(d, S = \pi_g)]$$

The following section address the measures that relate the saturation of treatment with the effects along the distribution of Y.

3 Spillover and saturation: a distributive analysis

The aim of this chapter is define and identify the direct and indirect effects for quantile. In addition to identifying these objects, the private and spillover effects for quantiles, similar to the average effects of the Philipson (2000), are also define from the proportion of treated.

The seminal paper on the direct and indirect treatment effects defined several concepts for the mean (Halloran & Struchiner 1991). Recently others authors analyzed some questions about average effects such as, for example, inference and properties of the estimators in large sample (Basse & Feller 2017, Hudgens & Halloran 2008, Vazquez-Bare 2017). However, despite the advances, effects along the distribution have not yet been analyzed.

The principal contribution of this work to the literature is the formalization of direct and indirect effects for quantile and also to present an estimator that allows to capture the effects of saturation over the Y distribution separating them into private and spillover effects similar to Philipson (2000).

In general, let $F_Y = \mathbb{P}(Y \leq y)$ and $F_{Y|X} = \mathbb{P}(Y \leq y|X)$ be the cumulative, unconditional and conditional respectively, distributions of $Y \in \mathbb{R}$, where X is any random variable. Let τ a real number where $\tau \in (0, 1)$, then the $\tau^t h$ quantile and the conditional quantile of Y is defined respectively as $q_\tau = F_Y^{-1}(\tau) = \inf_q\{\mathbb{P}[Y \leq q]\} \geq \tau$ and $q_{\tau|X} = F_{Y|X}^{-1}(\tau) = \inf_q\{\mathbb{P}[Y \leq q\ |X]\} \geq \tau$.

Thus, we can write the quantile, unconditional and conditional, at saturation level $S = \pi_g$ for treated and controls as:

$$q_{d,\pi_g,\tau} = \inf_q \left\{\mathbb{P}[Y_{ig}^{**}(d, S = \pi_g) \leq q]\right\} \geq \tau \tag{7}$$

$$q_{d,\pi_g,\tau|S=\pi_g} = \inf_q \left\{ \mathbb{P}[Y_{ig}^{**}(d, S=\pi_g) \leq q | S=\pi_g] \right\} \geq \tau \tag{8}$$

From the equation (8) some objects of interest can be written. For $S = \pi_g > 0$ and $S = \pi_g' \geq 0$ we defined the **Quantile Direct Effect (QDE)**, **Quantile Indirect Effect (QIE)** and **Quantile Saturation Effects (QSE)** as follows:

$$\varphi^D_{\pi_g,\tau} = q_{1,\pi_g,\tau|S=\pi_g} - q_{0,\pi_g,\tau|S=\pi_g} \tag{9}$$

$$\varphi^I_{\pi_g,\tau} = q_{0,\pi_g,\tau|S=\pi_g} - q_{0,0,\tau|S=0} \tag{10}$$

$$\varphi^S_{d,\pi_g,\pi_g',\tau} = q_{d,\pi_g,\tau|S=\pi_g} - q_{d,\pi_g',\tau|S=\pi_g'} \tag{11}$$

Note that the φ effects compares the quantile of the cumulative distribution between the groups for a given fixed percentile. We can calculate this objects for each $\tau \in (0,1)$ by obtaining a curve describing the effects for the entire distribution.

Comparisons can be made using several levels of saturation providing a better understanding of spillover effects throughout the distribution. Figure 1 is illustrative and aims to exemplify the types of analysis that can be made from these definitions. Suppose there is a RSD experiment with three saturated groups at the following levels: $(\pi_1, \pi_2, \pi_3) = (0, > 0, > 0)$. Before treatment the individuals are identical in observable and unobservable characteristics and after the treatment we observe the distributions in the figure 1. The graphs show the cumulative distribution for the control group resulting from null saturation ($\pi_1 = 0$), the control and treatment group without spillover and positive saturation ($\pi_2 > 0$), and the group with the same effects in π_2, but with spillover in the left tail ($\pi_3 > 0$). Note that the parts of the distributions may be different and the quantile direct, indirect and spillover effects will be distinct depending on the control group analyzed.

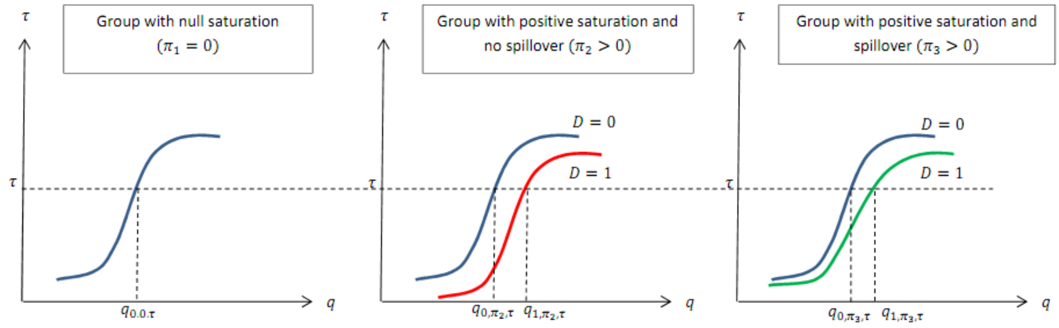

Figure 1: Cumulative distribution to three levels of saturation

Suppose that a researcher is interested in the effect of saturation on the individuals located in the percentile $\tau = 0.5$. In estimating the previously defined objects, it makes no difference to use the control group from the first distribution, $\pi_1 = 0$, or the second distribution, $\pi_2 > 0$, since they were not affected by treatment group. Thus, the indirect effect will be null when the control groups are from the saturation distributions in the levels (π_1, π_2).

The interesting case arises when comparing the distributions without and with spillover (π_2, π_3). Note that it is not possible to identify, a priori, how spillover affects the treatment and control groups in the last distribution, since an interaction may occur that benefits (or not) individuals in both groups. Thus, one can measure the effect of the treatment within the last distribution and also compare the quantiles among the distributions. To verify the effect on $q_{d,\pi_g,\tau=0,5}$, we can conditioning equations (9), (10) and (11) in the several saturation levels.

The last step to obtain the objects QDE, QIE and QSE is to show how we can write the quantiles as a function of the observed data coming from the random experiment. To identify these effects it will be necessary to assume an additional assumption:

Assumption 3 (A3: independence of potential outcomes)**.** *The potential outcomes are independent of treatment allocation and saturation level*

$$Y_{ig}^{**}(1, S = \pi_g), Y_{ig}^{**}(0, S = \pi_g) \perp (D_{ig}, \pi_g)$$

The assumption **A3** will be satisfied if the random experiment is conducted on two level as described in the previous section. Finally, the assumption **A1-A3** allows to rewrite the above definitions in function of the observed variables and of the sample analogues, which can be seen in lemma 1.

Lemma 1 (Identification of quantiles)**.** *Under assumptions **A1-A3** the quantiles can be written as:*

$$\tau = \mathbb{E}\left[\frac{D_{ig}}{\pi_g}\frac{\mathbb{I}\{S = \pi_g\}}{\mathbb{P}(S = \pi_g)}\mathbb{I}\{Y_{ig} \leq q_{1,\pi_g,\tau|S=\pi_g}\}\right]$$

$$\tau = \mathbb{E}\left[\frac{(1 - D_{ig})}{(1 - \pi_g)}\frac{\mathbb{I}\{S = \pi_g\}}{\mathbb{P}(S = \pi_g)}\mathbb{I}\{Y_{ig} \leq q_{0,\pi_g,\tau|S=\pi_g}\}\right]$$

$$\tau = \mathbb{E}\left[(1 - D_{ig})\frac{\mathbb{I}\{S = 0\}}{\mathbb{P}(S = 0)}\mathbb{I}\{Y_{ig} \leq q_{0,0,\tau|S=0}\}\right]$$

Proof. See appendix □

From this lemma φ effects can be written conditioning in the groups and treatment status to then obtain the quantiles:

Corollary 1 (Identification of QDE, QIE and QSE)**.** *Under assumptions **A1-A3** the quantiles effects, $\varphi^D_{\pi_g,\tau}$, $\varphi^I_{\pi_g,\tau}$, $\varphi^S_{d,\pi_g,\pi'_g,\tau}$, are identified.*

Proof. The proof is direct since they are functional of the data. □

Using the QDE, QIE and QSE it is possible to capture the direct and indirect effects of the treatment, but not the effects of the change in distribution due to saturation. The seminal paper of Philipson (2000) proposes to divide the changes in the distribution of Y, more specifically $\mathbb{E}(Y_{ig}|d, S = \pi_g)$, into two parts: i) a part from changes in distribution due to proportions of treated denominated "external effects" (or spillover) and ii) a second part attributed to treatment status named as "private effects".

To capture the private and spillover effects it is necessary to use the law of total expectation and partition the sample space as follows[2]: $\mathbb{E}(Y_i) = \pi\mathbb{E}(Y_i|D_i = 1) + (1 - \pi)\mathbb{E}(Y_i|D_i = 0)$. Deriving with respect to π: $\frac{d\mathbb{E}(Y)}{d\pi} = \mathbb{E}(Y_i|D_i = 1) - \mathbb{E}(Y_i|D_i = 0)$. Note that there is only private effects, because the expectation is not conditioned in the proportion of treated. So, to get both effects it is necessary rewriting the mean as: $\mathbb{E}(Y_{ig}|\pi_g) = \pi_g\mathbb{E}(Y_{ig}|S = \pi_g, D_{ig} = 1) + (1 - \pi_g)\mathbb{E}(Y_{ig}|S = \pi_g, D_{ig} = 0)$. Deriving gives the results of Philipson (2000, p.5).

$$\begin{aligned}\frac{d\mathbb{E}(Y_{ig}|\pi_g)}{d\pi_g} &= [\mathbb{E}(Y_{ig}|S = \pi_g, D_{ig} = 1) - \mathbb{E}(Y_{ig}|S = \pi_g, D_{ig} = 0)]+ \\ &+ \left[\pi_g \frac{d\mathbb{E}(Y_{ig}|D_{ig} = 1)}{d\pi_g} + (1 - \pi_g)\frac{d\mathbb{E}(Y_{ig}|D_{ig} = 0)}{d\pi_g}\right]\end{aligned}$$

In the case of quantiles it is necessary rewriting the cumulative distribution function by weighting for the proportion of treated individuals in each group:

$$\begin{aligned}F_{Y|S=\pi_g}(q_{\pi_g}) = \pi_g\mathbb{P}[Y^{**}_{ig}(1, S = \pi_g) \leq q_{\pi_g}|S = \pi_g, D_{ig} = 1]+ \\ + (1 - \pi_g)\mathbb{P}[Y^{**}_{ig}(0, S = \pi_g) \leq q_{\pi_g}|S = \pi_g, D_{ig} = 0]\end{aligned} \tag{12}$$

The following lemma shows the effect of π_g on the unconditional quantile q_{π_g}:

Lemma 2 (Saturation effects on the unconditional quantile)**.** *Under assumptions **A1-A2***

[2]The law of total expectation states that if $A_1, ..., A_n$ are partition of the sample space Ω then $\mathbb{E}(Y) = \sum_{i=1}^{n} \mathbb{E}(Y|A_i)\mathbb{P}(A_i)$. For quantiles, discussed below, we can use the law of total probability given by $\mathbb{P}(Y_{ig} \leq q_{\pi_g}) = \mathbb{P}(\{Y_{ig} \leq q_\pi\} \cap A) + \mathbb{P}(\{Y_{ig} \leq q_\pi\} \cap A^c) = \mathbb{P}(A)\mathbb{P}(Y_{ig} \leq q_\pi|A) + \mathbb{P}(A^c)\mathbb{P}(Y_{ig} \leq q_\pi|A^c)$. Since the expectation is conditional on two variables, π and D, it will also be used: $\mathbb{P}(Y|A) = \sum_n \mathbb{P}(Y|A \cap B)\mathbb{P}(A|B) = \sum_n \mathbb{P}(Y|A \cap B)\mathbb{P}(B)$ where B is an independent partition of A.

the saturation effects on the unconditional quantile can be written as:

$$
\begin{aligned}
\frac{dq_{\pi_g}}{d\pi_g} = \frac{-1}{f_Y(q_{\pi_g})} \Bigg[& \overbrace{\mathbb{P}[Y_{ig} \leq q_{\pi_g} | S = \pi_g, D_{ig} = 1] - \mathbb{P}[Y_{ig} \leq q_{\pi_g} | S = \pi_g, D_{ig} = 0]}^{\textit{Quantile Private Effect (QPE)}} + \\
& + \underbrace{\pi_g \frac{\partial \mathbb{P}[Y_{ig} \leq q_{\pi_g} | S = \pi_g, D_{ig} = 1]}{\partial \pi_g} + (1 - \pi_g) \frac{\partial \mathbb{P}[Y_{ig} \leq q_{\pi_g} | S = \pi_g, D_{ig} = 0]}{\partial \pi_g}}_{\textit{Quantile Spillover Effect (QSPE)}} \Bigg]
\end{aligned}
\tag{13}
$$

Proof. See appendix □

It can be seen that the relationship between the proportion of treated and the quantile depends on two parts called the Quantile Private Effect (QPE), difference between the quantiles of treated and controls given the saturation level, and the Quantile Spillover Effect (QSPE), defined by the relation between the derivatives and the saturation level, that is, the change in the distribution due to saturation. It should be noted that the expansion of the program, in this case, does not necessarily increase or reduce q_{π_g}, since the final variation will depend on the sizes of QPE, QSPE and also the density of Y valued in q_{π_g}. The effects of saturation on each quantile may be different because, depending on type of program and the interaction, the left tail, center and right tail of the distribution may be unequally affected.

Analyzing the equation (13) it is possible to identify the objects of interest with randomization, since the variation between groups generates QSPE (note that if there is no variability between each g, then the derivatives with respect to π_g will be null) and the variation within each group, randomization of treatment, allow get the QPE. If saturation is 100%, that is, $\pi_g = 1$, then, as expected, the quantile depend only on the private effects. Suppose that the $QPE > 0$ and $QSPE > 0$, that is, private effects and spillover effects positive, then $\frac{dq_{\pi_g}}{d\pi_g} > 0 \Leftrightarrow QPE < -QSPE$.

From the equation (13) several comparisons can be made by fixing a quantile and then conditioning the desire values of saturation and treatment. After presenting and interpreting the objects of interest, it remains discuss how to estimate the equations above.

4 Final considerations

The aim of this work is to define and identify the relationship of the saturation of the treatment with the quantiles of the outcome variable in the presence of treatment spillover. The literature has developed several estimators to obtain the effect of treatment with and without the presence of spillover for the mean, but there are still no work for quantile and treatment saturation.

As a continuation of this work we can develop the asymptotic theory as well as make inference and empirical estimates by checking the performance of the estimators. In addition, one can also remove the assumptions related to randomization and rewrite the measures using some kind of weighting to correct the treatment selection.

Finally, this work becomes invalid if there is selection in observable or unobservable, if the program induces network formation or even if not all treated unit are compliers.

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Proofs

A Identification proofs

Proof. Proof of lemma 1

Identification of the quantile for: $q_{1,\pi_g,\tau|S=\pi_g}$

The definition of τ conditional to the group with saturation level $S = \pi_g$ is given by:

$$\tau = \mathbb{P}\big(Y^{**}_{ig}(1, S = \pi_g) \leq q_{1,\pi_g,\tau|S=\pi_g}\big|S = \pi_g\big)$$

Under the assumption **A3** (independence):

$$\tau = \mathbb{P}\big(Y^{**}_{ig}(1, S = \pi_g) \leq q_{1,\pi_g,\tau|S=\pi_g}\big|S = \pi_g, D_{ig} = 1\big)$$

Replacing $Y_{ig} = Y^{**}_{ig}(1, S = \pi_g)D_{ig} + Y^{**}_{ig}(0, S = \pi_g)(1 - D_{ig})$:

$$\tau = \mathbb{P}\big(Y_{ig} \leq q_{1,\pi_g,\tau|S=\pi_g}\big|S = \pi_g, D_{ig} = 1\big)$$

Knowing that $\mathbb{P}(A) = \mathbb{E}[\mathbb{I}\{A\}]$. When replacing it multiplies by D_{ig} (because it is conditional on $D_{ig} = 1$) and $\mathbb{I}\{S = \pi_g\}$ (because it is conditional in the group $S = \pi_g$):

$$\tau = \mathbb{E}\big[D_{ig}\mathbb{I}\{S = \pi_g\}\mathbb{I}\{Y_{ig} \leq q_{1,\pi_g,\tau|S=\pi_g}\}\big|S = \pi_g, D_{ig} = 1\big]$$

Under the law of total probability:

$$\mathbb{E}[Y|S = \pi_g] = \pi_g\mathbb{E}[Y|S = \pi_g, D_{ig} = 1] + (1 - \pi_g)\mathbb{E}[Y|S = \pi_g, D_{ig} = 0]$$
$$\xrightarrow{D_{ig}=1} \mathbb{E}[Y|S = \pi_g, D_{ig} = 1] = \frac{\mathbb{E}[Y|S = \pi_g]}{\pi_g}$$

Rewriting

$$\tau = \mathbb{E}\left[\frac{D_{ig}}{\pi_g}\mathbb{I}\{S = \pi_g\}\mathbb{I}\{Y_{ig} \leq q_{1,\pi_g,\tau|S=\pi_g}\}\middle|S = \pi_g\right]$$

Applying the law of iterated expectations: $\mathbb{E}[\mathbb{E}[Y|X]] = p_1\mathbb{E}[Y|x = c_1] + p_2\mathbb{E}[Y|x = c_2] + ... + p_M\mathbb{E}[Y|x = c_M] = \mathbb{E}[Y] \xrightarrow{x=c_1} \mathbb{E}[Y|x = c_1] = \frac{\mathbb{E}[Y]}{p_1}$

$$\tau = \mathbb{E}\left[\frac{D_{ig}}{\pi_g}\frac{\mathbb{I}\{S = \pi_g\}}{\mathbb{P}(S = \pi_g)}\mathbb{I}\{Y_{ig} \leq q_{1,\pi_g,\tau|S=\pi_g}\}\right]$$

Identification of the quantile for: $q_{0,\pi_g,\tau|S=\pi_g}$

The definition of τ conditional to the group with saturation level $S = \pi_g$ is given by:

$$\tau = \mathbb{P}\big(Y^{**}_{ig}(0, S = \pi_g) \leq q_{0,\pi_g,\tau|S=\pi_g}\big|S = \pi_g\big)$$

Under the assumption **A3** (independence):

$$\tau = \mathbb{P}\big(Y^{**}_{ig}(0, S = \pi_g) \leq q_{0,\pi_g,\tau|S=\pi_g}\big|S = \pi_g, D_{ig} = 0\big)$$

Replacing $Y_{ig} = Y^{**}_{ig}(1, S = \pi_g)D_{ig} + Y^{**}_{ig}(0, S = \pi_g)(1 - D_{ig})$:

$$\tau = \mathbb{P}\big(Y_{ig} \leq q_{0,\pi_g,\tau|S=\pi_g}\big|S = \pi_g, D_{ig} = 0\big)$$

Knowing that $\mathbb{P}(A) = \mathbb{E}[\mathbb{I}\{A\}]$. When replacing it multiplies by $(1 - D_{ig})$ (because it is conditional on $D_{ig} = 0$) and $\mathbb{I}\{S = \pi_g\}$ (because it is conditional in the group $S = \pi_g$):

$$\tau = \mathbb{E}\big[(1 - D_{ig})\mathbb{I}\{S = \pi_g\}\mathbb{I}\{Y_{ig} \leq q_{0,\pi_g,\tau|S=\pi_g}\}\big|S = \pi_g, D_{ig} = 0\big]$$

Under the law of total probability:

$$\mathbb{E}[Y|S = \pi_g] = \pi_g\mathbb{E}[Y|S = \pi_g, D_{ig} = 1] + (1 - \pi_g)[Y|S = \pi_g, D_{ig} = 0]$$
$$\xrightarrow{D_{ig}=0} \mathbb{E}[Y|S = \pi_g, D_{ig} = 0] = \frac{\mathbb{E}[Y|S = \pi_g]}{(1 - \pi_g)}$$

Rewriting

$$\tau = \mathbb{E}\left[\frac{(1 - D_{ig})}{(1 - \pi_g)}\mathbb{I}\{S = \pi_g\}\mathbb{I}\{Y_{ig} \leq q_{1,\pi_g,\tau|S=\pi_g}\}\middle|S = \pi_g\right]$$

Applying the law of iterated expectations: $\mathbb{E}[\mathbb{E}[Y|X]] = p_1\mathbb{E}[Y|x = c_1] + p_2\mathbb{E}[Y|x = c_2] + ... + p_M\mathbb{E}[Y|x = c_M] = \mathbb{E}[Y] \xrightarrow{x=c_1} \mathbb{E}[Y|x = c_1] = \frac{\mathbb{E}[Y]}{p_1}$

$$\tau = \mathbb{E}\left[\frac{(1 - D_{ig})}{(1 - \pi_g)}\frac{\mathbb{I}\{S = \pi_g\}}{\mathbb{P}(S = \pi_g)}\mathbb{I}\{Y_{ig} \leq q_{1,\pi_g,\tau|S=\pi_g}\}\right]$$

Identification of the quantile for: $q_{0,0,\tau|S=0}$

The definition of τ conditional to the group with saturation level $S = 0$ is given by:

$$\tau = \mathbb{P}\big(Y^{**}_{ig}(0, S = 0) \leq q_{0,0,\tau|S=0}\big|S = 0\big)$$

Under the assumption **A3** (independence):

$$\tau = \mathbb{P}\big(Y^{**}_{ig}(0, S = 0) \leq q_{0,0,\tau|S=0}\big|S = 0, D_{ig} = 0\big)$$

Replacing $Y_{ig} = Y^{**}_{ig}(1, S = \pi_g)D_{ig} + Y^{**}_{ig}(0, S = \pi_g)(1 - D_{ig})$:

$$\tau = \mathbb{P}\big(Y_{ig} \leq q_{0,0,\tau|S=0}\big|S = 0, D_{ig} = 0\big)$$

Knowing that $\mathbb{P}(A) = \mathbb{E}[\mathbb{I}\{A\}]$. When replacing it multiplies by $(1 - D_{ig})$ (because it is conditional on $D_{ig} = 0$) and $\mathbb{I}\{S = 0\}$ (because it is conditional in the group $S = 0$):

$$\tau = \mathbb{E}\big[(1 - D_{ig})\mathbb{I}\{S = 0\}\mathbb{I}\{Y_{ig} \leq q_{0,0,\tau|S=0}\}\big|S = 0, D_{ig} = 0\big]$$

Under the law of total probability:

$$\mathbb{E}[Y|S=\pi_g] = \pi_g \mathbb{E}[Y|S=\pi_g, D_{ig}=1] + (1-\pi_g)[Y|S=\pi_g, D_{ig}=0]$$
$$\xrightarrow{D_{ig}=0} \mathbb{E}[Y|S=\pi_g, D_{ig}=0] = \frac{\mathbb{E}[Y|S=\pi_g]}{(1-\pi_g)}$$

Rewriting

$$\tau = \mathbb{E}\Big[(1-D_{ig})\mathbb{I}\{S=0\}\mathbb{I}\{Y_{ig} \leq q_{0,0,\tau|S=0}\}\Big|S=\pi_g\Big]$$

Applying the law of iterated expectations: $\mathbb{E}[\mathbb{E}[Y|X]] = p_1\mathbb{E}[Y|x=c_1] + p_2\mathbb{E}[Y|x=c_2] + ... + p_M\mathbb{E}[Y|x=c_M] = \mathbb{E}[Y] \xrightarrow{x=c_1} \mathbb{E}[Y|x=c_1] = \frac{\mathbb{E}[Y]}{p_1}$

$$\tau = \mathbb{E}\left[(1-D_{ig})\frac{\mathbb{I}\{S=0\}}{\mathbb{P}(S=0)}\mathbb{I}\{Y_{ig} \leq q_{0,0,\tau|S=0}\}\right]$$

□

B Unconditional quantile proof

In general, define $f(x, y(x)) = c \Rightarrow f(x, y(x)) - c = f^*(x, y(x)) = 0$. The following derivative $\frac{dy}{dx}$ can be calculated using the implicit function theorem:

$$\frac{\partial f(x,y(x))}{\partial x} + \frac{\partial f(x,y(x))}{\partial y}\frac{dy}{dx} - \frac{dc}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f(x,y(x))}{\partial x}}{\frac{\partial f(x,y(x))}{\partial y}} = -\frac{\frac{\partial f^*(x,y(x))}{\partial x}}{\frac{\partial f^*(x,y(x))}{\partial y}}$$

Proof. Saturation effects on the unconditional quantile

Rewriting equation (12):

$$\begin{aligned} F_{Y|S=\pi_g}(q_{\pi_g}) - \pi_g \mathbb{P}[Y^{**}_{ig}(1, S=\pi_g) \leq q_{\pi_g}|S=\pi_g, D_{ig}=1] - \\ - (1-\pi_g)\mathbb{P}[Y^{**}_{ig}(0, S=\pi_g) \leq q_{\pi_g}|S=\pi_g, D_{ig}=0] = \\ = F^*(\pi_g, q_{\pi_g}) = 0 \end{aligned} \tag{A1}$$

Applying the implicit function theorem in equation $(A1)$

$$
\begin{aligned}
\frac{dq_{\pi_g}}{d\pi_g} &= -\frac{\partial F^*(\pi_g, q_{\pi_g})/\partial \pi_g}{\partial F^*(\pi_g, q_{\pi_g})/\partial q_{\pi_g}} = \\
&= -\frac{1}{f_Y(q_{\pi_g})}\Bigg[-\mathbb{P}[Y^{**}_{ig}(1, S=\pi_g) \leq q_{\pi_g} | S = \pi_g, D_{ig} = 1] - \pi_g \frac{\partial \mathbb{P}[Y^{**}_{ig}(1, S=\pi_g) \leq q_{\pi_g} | S = \pi_g, D_{ig} = 1}{\partial \pi_g} \\
&- \frac{\partial \mathbb{P}[Y^{**}_{ig}(0, S=\pi_g) \leq q_{\pi_g} | S = \pi_g, D_{ig} = 0]}{\partial \pi_g} + \mathbb{P}[Y^{**}_{ig}(0, S=\pi_g) \leq q_{\pi_g} | S = \pi_g, D_{ig} = 0] + \\
&+ \pi_g \frac{\partial \mathbb{P}[Y^{**}_{ig}(0, S=\pi_g) \leq q_{\pi_g} | S = \pi_g, D_{ig} = 0]}{\partial \pi_g}\Bigg]
\end{aligned}
$$

Simplifying and replacing $Y_{ig} = Y^{**}_{ig}(1, S = \pi_g)D_{ig} + Y^{**}_{ig}(0, S = \pi_g)(1 - D_{ig})$:

$$
\begin{aligned}
\frac{dq_{\pi_g}}{d\pi_g} = \frac{-1}{f_Y(q_{\pi_g})}\Bigg[& \overbrace{\mathbb{P}[Y_{ig} \leq q_{\pi_g} | S = \pi_g, D_{ig} = 1] - \mathbb{P}[Y_{ig} \leq q_{\pi_g} | S = \pi_g, D_{ig} = 0]}^{\text{Quantile Private Effect (QPE)}} + \\
& + \underbrace{\pi_g \frac{\partial \mathbb{P}[Y_{ig} \leq q_{\pi_g} | S = \pi_g, D_{ig} = 1]}{\partial \pi_g} + (1 - \pi_g)\frac{\partial \mathbb{P}[Y_{ig} \leq q_{\pi_g} | S = \pi_g, D_{ig} = 0]}{\partial \pi_g}}_{\text{Quantile Spillover Effect (QSPE)}}\Bigg]
\end{aligned}
$$

□

The **A1-A2** assumptions have no explicit role in the proof. Such assumptions allow the potential outcome $Y_{ig}(d, S = \pi_g)$ to depend only on the treatment status and the saturation level.