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Quadratic Funding with Incomplete Information

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JEL Codes: C72, D82, H41

Quadratic Funding with Incomplete Information

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Abstract

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1 Introduction

Public goods provision is one of the problems in Public Economics that attracts a significant share of the attention of theorists and applied researchers alike. Non-excludability and non-rivalry of these kinds of goods make arising the well-known free-rider problem, which represents a challenge both for the theoretical and applied point of view. Samuelson (1954) was one of the pioneers in identifying and modeling the problem and proposing solutions. A classic solution for the problem is the Lindahl taxation (Lindahl (1958)); in it, the central planner (government) uses the private demand of each participant to determine the optimal level of funding and, charge participants according to their marginal willingness to pay. Evident difficulties for that solution are the definition of personalized markets for the good and confidence in individuals' truthful revelation of values.

Modern mechanism proposals include restrictions of participation and incentives to reveal the true valuation for the public good. Clarke (1971), Groves (1973), and Groves and Ledyard (1977) contributed with the formulation of government allocation-taxation schemes that recover the Pareto optimality, even in general equilibrium frameworks. Some criticisms have been made regarding that mechanism due to its weakness to collusions formation and other impractical issues (Rothkopf (2007)). Walker (1981)proposes a variation of this mechanism that manages to satisfy both efficiency and individual rationality. Nevertheless, the resulting

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mechanism is subject to the instability of equilibria, an essential property for its practical implementation (Healy (2006)). Tideman and Plassmann (2017) make a detailed discussion of mechanisms in this literature and their common characteristics.

Buterin et al. (2019) have recently proposed a new mechanism capable of attaining optimality in the private provision of a public good: the quadratic funding (QF) mechanism. It is based on the quadratic voting mechanism proposed by Weyl (Posner and Weyl (2017)), which is already applied to democratic politics and corporate governance. The authors claim that QF would not require any assumption about the set of public goods to be funded nor about the number of individuals contributing to these public goods. Such properties are not shared by other mechanisms in the literature and would make the mechanism particularly well-suited for situations where one would wish for the individuals themselves to propose public goods for funding, rather than letting these individuals choose contributions for a fixed set of public goods. The QF mechanism is already used in platforms for funding public goods and open-source projects, such as HackerLink and Gitcoin.

When models of public goods provision include the possibility of incomplete information regarding individual characteristics, it is often the case that inefficiency increases. Some works in the literature show different ways to include incomplete information in the public good provision model and its consequences. Menezes et al. (2001)and Bag and Roy (2008) consider the incomplete information in the participants' preferences. Gradstein (1992) and Gradstein et al. (1994) insert the incomplete information in the contributors' wealth. Finally, Maldonado and Rodrigues-Neto (2016) model the information incompleteness stemming from the anonymity of the players.

This work analyzes the effects of incomplete information in public goods provision games when the QF mechanism finances the provision of the public good. Due to the positive results of this sort of funding mechanisms when individuals have complete information and the conjecture proposed in Buterin et al. (2019) that those results would also be valid in incomplete information settings, we propose a model with those characteristics and study under which conditions the efficiency property remains. We show that the Pareto optimality of the private and decentralized provision is only satisfied under very restrictive conditions. We provide necessary and sufficient conditions to obtain efficiency, and we also propose two measures for the size of the inefficiency in that type of game.

This article contains five sections. After this introductory section, section 2 defines the public good provision game under complete information and with general funding mechanisms. We prove that QF is the unique mechanism providing efficiency among a class of funding mechanisms. In addition, we give necessary and sufficient conditions to have such efficiency with null or strictly positive amounts of the public good, complementing some conclusions obtained by Buterin et al. (2019). Section 3 extends the framework to the incomplete information case and proves necessary and sufficient conditions for efficiency in this setting. In particular, we prove that QF is efficient for individuals with CRRA utility functions for the public good if and only if the relative risk aversion coefficient is equal to $\frac{1}{2}$, a very restrictive condition. Given the large class of models where the inefficiency of private provision of the public good is present, section 4 proposes two measures for the size of the inefficiency (deadweight loss) of the equilibrium. We analyze the response of those measures to variations in the parameters related to the incomplete information and the number of participants; this is for assessing possible asymptotic efficiency. Finally, in section 5, we summarize some conclusions of the work, and in the Appendix, we provide the proofs of the stated propositions.

2 The quadratic funding mechanism

In this section, we present the quadratic funding mechanism (henceforth QF) for the provision of continuous public goods when the individuals own complete information regarding the preferences of the participants. We analyze properties of existence and efficiency of the solution of that mechanism. This will serve as a benchmark for the ensuing analysis under incomplete information.

There exist $n \geq 2$ individuals identified by the elements in the set $\mathcal{N} = \{1, 2, \cdots, n\}$ and a single public good to be provided. Each individual $i \in \mathcal{N}$ is defined by its quasilinear utility function $u_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined as $u_i(F, m) = v_i(F) + m$, where the linear good is the numeraire and $v_i : \mathbb{R}_+ \to \mathbb{R}$ represents the monetary-equivalent utility of consumption of the public good. We assume every function v_i is C^1, strictly increasing, and strictly concave. Moreover, we suppose that for every $i \in \mathcal{N}$, $\lim_{F\to\infty} v_i'(F) = 0$. As we show later, this hypothesis will guarantee the existence of an efficient public good provision level.[1]

Definition 2.1. *A **funding mechanism** is a function $\Phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ that determines, for any contribution profile of the individuals to the public good $\boldsymbol{c} := (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n_+$, a level of public good provision $\Phi(\boldsymbol{c}) = F \in \mathbb{R}_+$.*

In words, a funding mechanism is a technology that transforms individual contributions (inputs) into a level of the public good to be provided (output). The simple and classical funding mechanism is the linear technology $\Phi(\mathbf{c}) = \sum_{i=1}^n c_i$. Whenever possible, a central planner would choose a funding mechanism that fosters individual contributions that generate efficient output levels. However, before delving into efficiency, let us define the concept of a public good contribution game with a funding mechanism and its corresponding Nash equilibrium.

Definition 2.2. *A **public good provision game with a funding mechanism** Φ under complete information is defined by*

$$\mathcal{G} = \{(v_i)_{i\in\mathcal{N}}, \Phi\}.$$

Definition 2.3. *An allocation $\boldsymbol{c}^* \in \mathbb{R}^n_+$ is an **equilibrium** for $\mathcal{G}$ if, for all $i \in \mathcal{N}$, we have that $v_i(\Phi(c_i^*, \boldsymbol{c}_{-i}^*)) - c_i^* \geq v_i(\Phi(z, \boldsymbol{c}_{-i}^*)) - z$, for all $z \geq 0$. Equivalently, $c_i^* = \arg\max_{z\geq 0} v_i(\Phi(z, \boldsymbol{c}_{-i}^*)) - z$.*

*An equilibrium $\boldsymbol{c}^*$ is called **interior** if $c_i^* > 0$, for all $i \in \mathcal{N}$.*

Now, let us turn to the definition of efficiency, or optimality. The social welfare function in this context is the function assigning to each level of public good provision its social net value, namely,

$$W(F) = \left(\sum_{i=1}^n v_i(F)\right) - F.$$

From a normative point of view, a desirable property for a funding mechanism is the ability to generate private contributions that attain efficient levels of the public good. We formalize these notions in the following definitions.

[1] That hypothesis may be weakened. For example, if we assume instead that for all $i \in \mathcal{N}$ there exists $F \geq 0$ such that $v_i'(F) < 1/n$, the same conclusions can be obtained.

Definition 2.4. *A funding level $F^e \geq 0$ is said to be a* ***socially optimal provision****, or* ***efficient provision****, if*

$$F^e = \arg\max_{F \geq 0} \left(\sum_{i=1}^{n} v_i(F) \right) - F.$$

That is, F^e maximizes the total social welfare. The first order condition, which is necessary and sufficient in our setting, implies that $\sum_{i=1}^{n} v_i'(F^e) \leq 1$, with equality holding when $F^e > 0$.

Definition 2.5. *Let $\mathcal{G} = \{(v_i)_{i \in \mathcal{N}}, \Phi\}$. The funding mechanism Φ is* ***optimal****, or* ***efficient****, if there exists an equilibrium $\boldsymbol{c}^* \in \mathbb{R}^n_+$ for $\mathcal{G}$, such that $\Phi(\boldsymbol{c}^*)$ is a socially optimal provision.*

Definition 2.5 is our own, and it is slightly different from that stated by Buterin et al. (2019). Their definition of funding mechanism optimality is akin to definition 2.4, which is related to the optimality of the public good level, and not to the mechanism. Furthermore, it is not linked with the effect of the funding mechanism on individual decisions. More importantly, it excludes the possibility of a multiplicity of equilibria, which we will show to occur even for quadratic funding. For these reasons, definition 2.5 is more appropriate for mechanism optimality.

As previously discussed, the technology defining the funding mechanism Φ may be a general one with some suitable properties. Homogeneity of degree one ($\Phi(\lambda \mathbf{c}) = \lambda \Phi(\mathbf{c})$) guarantees the irrelevance of the units used to measure the contributions. Anonymity ($\Phi(\mathbf{c}) = \Phi(\sigma(\mathbf{c}))$, where $\sigma(\cdot)$ is the permutation operator), guarantees the irrelevance of the order of contributors. Inada's condition of the $i-$player when someone else is contributing with a strictly positive amount ($\lim_{c_i \to 0} \Phi_i(c_i, \mathbf{c}_{-i}) = +\infty$, where $\mathbf{c}_{-i} \neq 0$ and Φ_i is the partial derivative of Φ with respect to the ith component) stimulates individual i to contribute whenever other individuals are providing positive contributions to the public good. A technology owning all these properties is the CES technology $\Phi(\mathbf{c}) = [\sum_{i=1}^{n} c_i^{\rho}]^{1/\rho}$, with $\rho < 1$ to guarantee the strict convexity of the technology. To this family of technologies belongs the QF mechanism.

Definition 2.6. *The* ***quadratic funding mechanism*** *is defined by the function $\Phi^{QF} : \mathbb{R}^n_+ \to \mathbb{R}_+$ given by:*

$$\Phi^{QF}(\boldsymbol{c}) = \left(\sum_{i=1}^{n} (c_i)^{1/2} \right)^2.$$

In addition to satisfying the properties mentioned in the paragraph above, the QF mechanism was extensively discussed in Posner and Weyl (2017) as a quadratic voting rule, highlighting its efficiency properties and the concerns to achieve them. As a funding mechanism, we show in the following propositions that, among all the CES mechanisms, QF is the only one that provides an efficient public good level.

Firstly, let us prove that under very general hypotheses on the individuals' utilities we have a unique efficient level of the public good.

Proposition 2.1. *Suppose that, for all $i \in \mathcal{N}$, we have that $v_i \in C^1$ is a strictly concave function, and $\lim_{F \to \infty} v_i'(F) = 0$. Then, there exists a unique efficient provision $F^e \geq 0$.*

Before stating the result linking equilibria with QF and efficiency, let us show that interior equilibria generated by games with CES funding mechanisms are efficient only when the QF funding mechanism is used.

Proposition 2.2. *Let $\mathcal{G}$ be a game where for all $i \in \mathcal{N}$, $v_i \in C^1$ is an increasing and strictly concave function, and the funding mechanism is $\Phi(\boldsymbol{c}) = [\sum_{i=1}^n c_i^\rho]^{1/\rho}$, $\rho < 1$. If $\boldsymbol{c} \gg 0$ is an interior equilibrium of this mechanism and the efficient allocation is $F^e > 0$, then:*

(i) If $\rho > 1/2$, then $\Phi(\boldsymbol{c}) < F^e$;

(ii) If $\rho < 1/2$, then $\Phi(\boldsymbol{c}) > F^e$;

(iii) If $\rho = 1/2$, then $\Phi(\boldsymbol{c}) = F^e$.

The proposition above guarantees that, among the CES funding mechanisms, only QF generates efficient levels of the public good, provided that an equilibrium exists and the corresponding contributions are strictly positive. For this reason we will center our analysis on the QF mechanism. In the remainder of this section, we study existence and efficiency of equilibrium under this mechanism, and then conclude some additional properties of the equilibria when we restrict the analysis to CRRA utility functions for the public good.

The following proposition states existence and efficiency even in the case of corner solutions.

Proposition 2.3. *Suppose that, for all $i \in \mathcal{N}$, we have that $v_i \in C^1$ is a strictly increasing and strictly concave function, and $\lim_{F \to +\infty} v_i'(F) = 0$. Then, the quadratic funding mechanism is optimal.*

Despite the proposition 2.3 asserting that QF generates at least one equilibrium that provides an efficient level of the public good, it is important to highlight that it may also generate other inefficient equilibrium (an equilibrium providing an inefficient level of the public good). For example, consider the hypotheses of that proposition and suppose that (a) for all $i \in \mathcal{N}$ we have that $v_i'(0) \leq 1$ and (b) $\sum_{i=1}^n v_i'(0) > 1$. In such case, it is easy to check that (a) implies that $\mathbf{c}^* = 0$ is an equilibrium. On the other hand, under (b) there exists $F^e > 0$ satisfying $\sum_{i=1}^n v_i'(F^e) = 1$ and therefore, $F^e > 0$ is the efficient level. With that level we can define, for each $i \in \mathcal{N}$, $c_i^{**} = \left(v_i'(F^e) \cdot (F^e)^{1/2}\right)^2$ and easily verify that $\mathbf{c}^{**} = (c_1^{**}, \cdots, c_n^{**})$ is an interior equilibrium providing the efficient level of the public good. Thus, we can resume these analyses in the following proposition.

Proposition 2.4. *Suppose that, for all $i \in \mathcal{N}$, we have that $v_i \in C^1$ is a strictly increasing and strictly concave function, and $\lim_{F \to +\infty} v_i'(F) = 0$. Then, the quadratic funding mechanism has an inefficient equilibria if and only if for all $i \in \mathcal{N}$ we have that $v_i'(0) \leq 1$ and zero is not the efficient provision.*

To finalize this section, let us illustrate the case where the preferences of a participant is represented by a CRRA utility function. The reason for that is two-fold: first, we want to analyze the reaction of the best-response of an individual to variations of the contributions of all others, depending on the risk aversion parameter, and second, because these results will be useful for later analysis with incomplete information.

To be precise, let $i = 1$ index the individual with utility function for the public good $v_1(F) = \beta F^{1-\gamma}/(1-\gamma)$, if $\gamma \neq 1$ and $v_1(F) = \ln(F)$, if $\gamma = 1$. The first order condition defining c_1, the best-response of this individual to the contribution profile $\mathbf{c}_{-1}$ of all the other individuals is:

$$\frac{c_1^{1/2}}{\sum_{j=1}^n c_j^{1/2}} = \frac{\beta}{\left(\sum_{j=1}^n c_j^{1/2}\right)^{2\gamma}}. \tag{1}$$

The right-hand-side of (1) represents the marginal utility of increasing the provision of the public good in one (infinitesimal) unit, whereas the left-hand-side is the marginal increase of the individual contribution per unit of increase in the level of the public good.

To analyze the best-response variation to changes in the contributions of other individuals, let us rewrite (1) as follows:

$$c_1^{1/2} = \beta(c_1^{1/2} + \sum_{j\neq 1}^{n} c_j^{1/2})^{1-2\gamma}.$$

Thus, if $\gamma > 1/2$, then an increase in the aggregate square-roots of the others' contributions will produce a reduction in the best-response c_1. The reciprocal effect is obtained if $\gamma < 1/2$. Finally, if $\gamma = 1/2$, the best-response of individual $i = 1$ does not depend on the contributions of her peers.

To summarize the above analysis we state the following proposition.

Proposition 2.5. *Suppose that an individual has a CRRA utility function for the public good with relative risk aversion coefficient $\gamma > 0$. Then, the individual's contribution to the public good is a decreasing function of the aggregate square-roots of the other individuals' contributions if, and only if, $\gamma \geq 1/2$.*

An interesting conclusion that results from proposition 2.5 is the stabilizer response of an individual with constant relative risk aversion to variations in the contributions of her peers. If she is highly risk averse ($\gamma > 1/2$), decreases in the others' contributions lead her to contribute more. On the other hand, with a low risk aversion ($\gamma < 1/2$) she diminishes her own contribution when she perceives that the other participants diminish theirs. In the limit case ($\gamma = 1/2$) , she is indifferent to variations of others' contributions.

3 Quadratic funding under incomplete information

In this section we will extend the public good contribution game with funding mechanism framework to the case where participants have incomplete information regarding their preferences for the good. We prove existence and efficiency of equilibria and that the conditions for efficiency under the quadratic funding mechanism are strong, that is, efficiency under QF is difficult to attain under incomplete information.

As in section 2 we have a set $\mathcal{N}$ of individuals. For each $i \in \mathcal{N}$ there exists a finite set of types $\Theta_i = \{\theta_i^\ell; 1 \leq \ell \leq L_i, L_i \in \mathbb{N}\}$ associated and an expected utility function $u_i : \mathbb{R}_+ \times \mathbb{R} \times \Theta_i \to \mathbb{R}$ defined as $u_i(F, m; \theta_i) = v_i(F; \theta_i) + m$, where the linear good is the numeraire and $v_i : \mathbb{R}_+ \times \Theta_i \to \mathbb{R}$ represents the monetary-equivalent expected utility of i for a level $F \geq 0$ of funding for the public good when her type is $\theta_i \in \Theta_i$. We use $\Theta = \times_{i=1}^n \Theta_i$ as the set of all possible states of the world. The joint probability distribution of types is $\Pr : \Theta \to [0, 1]$, which is assumed to be common knowledge. As in the complete information case, we assume that, for every $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$, $v_i(\cdot; \theta_i) \in C^1$ is a strictly increasing and strictly concave function, and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. With all these elements we can define the framework under incomplete information.

Definition 3.1. *A* ***public good provision game with funding mechanism*** Φ *and* ***incomplete information*** *is defined by*

$$\mathcal{G} = \left\{ (v_i, \Theta_i)_{i\in\mathcal{N}}, \Pr, \Phi \right\}.$$

For that kind of game, the Bayes-Nash equilibrium concept is as follows.

Definition 3.2. *A profile* $\boldsymbol{c}^* = (c_1^*, \cdots, c_n^*)$*, where for all* $i \in \mathcal{N}$*,* $c_i^* : \Theta_i \to \mathbb{R}_+$ *is an* ***equilibrium*** *for* $\mathcal{G}$ *if, for each* $i \in \mathcal{N}$ *and each* $\theta_i \in \Theta_i$*, the following is satisfied for all* $z \in \mathbb{R}_+$ *:*

$$E\left[v_i(\Phi(c_i^*(\theta_i), \boldsymbol{c}_{-i}^*(\theta_{-i})); \theta_i) \,\middle|\, \theta_i \right] - c_i^*(\theta_i) \geq E\left[v_i(\Phi(z, \boldsymbol{c}_{-i}^*(\theta_{-i})); \theta_i) \,\middle|\, \theta_i \right] - z.$$

Alternatively, for all $i \in \mathcal{N}$ *and all* $\theta_i \in \Theta_i$*,* $\boldsymbol{c}^*$ *satisfies:*

$$c_i^*(\theta_i) = \arg\max_{z\in\mathbb{R}_+} E\left[v_i(\Phi(z, \boldsymbol{c}_{-i}^*(\theta_{-i})); \theta_i) \,\middle|\, \theta_i \right] - z.$$

To deal with the efficiency concept, we will adopt the *ex-post efficiency* idea, where the central planner is maximizing the welfare function $W(F;\theta) = \sum_{i=1}^n v_i(F;\theta_i) - F$, for each type profile $\theta \in \Theta$. Then we propose the next definition.

Definition 3.3. *We say that* $F^e : \Theta \to \mathbb{R}_+$ *is an* ***(ex-post) efficient provision*** *for* $\mathcal{G}$ *if, for all* $\theta \in \Theta$*,*

$$F^e(\theta) = \arg\max_{F\geq 0} \left(\sum_{i=1}^n v_i\left(F;\theta_i\right) \right) - F.$$

It follows from the first order conditions that, for all $\theta \in \Theta$*, we have* $\sum_{i=1}^n v_i'(F^e(\theta);\theta_i) \leq 1$*, with equality if* $F^e(\theta) > 0$*.*

As a direct extension of definition 2.5 to the incomplete information framework, we have the following definition.

Definition 3.4. *The funding mechanism* Φ *is* ***optimal****, or* ***efficient*** *for* $\mathcal{G} = \left\{ (v_i, \Theta_i)_{i\in\mathcal{N}}, \Pr, \Phi \right\}$ *if there exists an equilibrium contribution profile* $\boldsymbol{c}^*$ *such that, for all* $\theta \in \Theta$*,* $\Phi(\boldsymbol{c}^*(\theta)) = F^e(\theta)$ *is an efficient provision for* $\mathcal{G}$*.*

Since QF was the only mechanism producing efficient interior equilibria under complete information among all CES mechanisms, we will restrict our analysis to this funding mechanism under incomplete information and study in which cases the same property is preserved. The QF mechanism is analogously defined in this context and in order to simplify notations we use $F(\theta) := \Phi^{QF}(\mathbf{c}(\theta))$ to denote the funding provided by QF when players choose the strategy profile $\mathbf{c}(\theta)$ and $F^*(\theta) := \Phi^{QF}(\mathbf{c}^*(\theta))$ if $\mathbf{c}^*$ is an equilibrium for QF.

Before stating our first results in this framework, let us analyze the level of the public good provided by QF in the case of interior equilibria. Thus, let $\mathbf{c}^*$ be an interior equilibrium for the QF mechanism ($\mathbf{c}^*(\theta) \gg 0$ for all $\theta \in \Theta$). The first order conditions imply that for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$ we must have

$$(c_i^*(\theta_i))^{1/2} = E\left[v_i'(F^*(\theta)); \theta_i) \cdot (F^*(\theta))^{1/2} \,\middle|\, \theta_i \right]. \tag{2}$$

Taking the expected value in (2), making the summation and rearranging the terms, we obtain:

$$E\left[\left[\left(\sum_{i=1}^{n} v_i'(F^*(\theta));\theta_i)\right)-1\right]\cdot (F^*(\theta))^{1/2}\right]=0. \tag{3}$$

Notice that equation (3) does not imply that the term in the inside brackets must be zero (which corresponds to the case of efficient provision of the public good). Indeed, as we show later, this implication does not follow. It is also important to highlight that in the complete information framework the same problem does not appear since there is no expected value and the term in the inside brackets is always null for positive funding of the public good.

Our first result is the existence and uniqueness of efficient provision.

Proposition 3.1. *Suppose that, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i) \in C^1$ is a strictly concave function and $\lim_{F\to\infty} v_i'(F;\theta_i)=0$. Then, there exists a unique efficient provision $F^e:\Theta\to\mathbb{R}_+$.*

To prove the existence of equilibrium of games with QF mechanisms, we will first prove that the best-response correspondence is at most single valued, and that there exists a stable domain where the best-response functions are defined.

Lemma 3.1. *Suppose that, for all $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i) \in C^1$ is a strictly increasing and strictly concave function. Then, for any $i \in \mathcal{N}$ and $\boldsymbol{c}_{-i} \in \mathbb{R}_+^{n-1}$, the best-response correspondence $c_i(\cdot;\boldsymbol{c}_{-i}):\Theta_i\to\mathbb{R}_+$ is at most single valued.*

Lemma 3.2. *Suppose that, for all $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i) \in C^1$ is a strictly concave function and $\lim_{F\to\infty} v_i'(F;\theta_i)=0$. Then, there exists $A>0$ such that, for any $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$, if the contributions of all other individuals belong to $[0,A]$ for any profile of types, then i's best-response also belongs to that interval.*

Now, we can state, under very classical assumptions on fundamentals, the existence of equilibrium for the game with incomplete information.

Proposition 3.2. *Suppose that, for all $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i) \in C^1$ is a strictly increasing and strictly concave function, and $\lim_{F\to\infty} v_i'(F;\theta_i)=0$. Then, there exists an equilibrium for the quadratic funding mechanism.*

Despite the existence of equilibrium for QF under general conditions on fundamentals, efficiency is rarely fulfilled. Unlike efficiency under complete information, when individuals are uncertain of the preferences of their peers, efficiency only results under strong conditions on the utility functions. In the sequel, we will establish a number of necessary and sufficient conditions that guarantee this property.

First, let us consider scenarios where not providing the public good might be efficient.

Proposition 3.3. *Suppose that for all $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$ the function $v_i(\cdot;\theta_i) \in C^1$ is a strictly increasing and strictly concave function, and $\lim_{F\to\infty} v_i'(F;\theta_i)=0$. Additionally, suppose that there exists some $\theta' \in \Theta$ such that $F^e(\theta')=0$, and that $\Pr(\theta)>0$ for all $\theta \in \Theta$. Then, the quadratic funding mechanism is efficient if and only if $F^e(\theta)=0$ for all $\theta \in \Theta$.*

Proposition 3.3 does not have a counterpart in the complete information setting and shows that there exists a broad range of public good games with quadratic funding where the equilibrium is not efficient. It suffices to exist two profiles, one with zero efficient provision and another with positive efficient provision, to have inefficiency of the equilibrium. Thus, in order to look for the cases where efficiency of QF is obtained, let us consider games where the efficient level of the public good is strictly positive and try to find conditions on the preferences that allows for the efficiency of the equilibrium. The following proposition states those conditions.

Proposition 3.4. *Suppose that for all $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$ the function $v_i(\cdot;\theta_i) \in C^1$ is strictly increasing and strictly concave, and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. Additionally, suppose that $F^e(\theta) > 0$, for all $\theta \in \Theta$ and there is only one individual with more than one type; namely, there exists $j \in \mathcal{N}$ such that $|\Theta_j| > 1$ and $\Theta_i = \{\theta_i^1\}$ for all $i \neq j$. Then, the quadratic funding mechanism is efficient if, and only if, there exists $A \in \mathbb{R}$ such that $\sum_{i\neq j} v_i'(F^e(\theta);\theta_i^1) = A(F^e(\theta))^{-1/2}$ for all $\theta \in \Theta$. Furthermore, we have that $A = \sum_{i\neq j}(c_i^*(\theta_i^1))^{1/2}$.*

In words, in the simple case of only one individual having several types and the efficient provision level always being positive, a necessary and sufficient condition for having an efficient equilibrium for QF is that the aggregate marginal utility of all other individuals is similar to a marginal CRRA utility function with risk aversion coefficient equal to 1/2, when evaluated in the efficient level of public good provision. That gives us a clue about where we can look for utility functions that allows for efficiency of equilibrium.

It is possible to obtain utility functions with diverse formats that produce an efficient equilibrium. For example, suppose that $n = 3$, the utility functions of individuals 2 and 3 are $v_2(F;\theta_2^1) = F^{1/2} + (F+1)^{1/6}$ and $v_3(F;\theta_3^1) = F^{1/2} - (F+1)^{1/6}$, and let $v_1(\cdot;\theta_1)$ be a function satisfying the hypotheses of proposition 3.4. Both v_2 and v_3 are strictly increasing and strictly concave functions. Since $v_2'(F;\theta_2^1) + v_3'(F;\theta_3^1) = 4F^{-1/2}$, proposition 3.4 guarantees the efficiency of QF , no matter what the function v_1 is and the number of types it could have.

To finalize this section, we state the result that shows the strong conditions under which the equilibrium of quadratic funding is efficient for the class of CRRA utility functions representing the preferences of the participants. It is worth noting the contrast with the case of the complete information framework. In proposition 2.2 we had that, with very general conditions on the utility functions of the contributors, the efficiency of the equilibrium is guaranteed. However, even though the contributors have CRRA utility functions, the uncertainty regarding the preferences makes each player contribute amounts that, when aggregated by the funding mechanism, are not efficient, except in the special case where every individual has a risk aversion coefficient equal to 1/2.

Proposition 3.5. *Suppose that for every $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$, we have that $v_i(F;\theta_i) = \beta_i(\theta_i)F^{1-\gamma}/(1-\gamma)$, where $\beta_i(\theta_i) > 0$ and $\gamma > 0$. Additionally, suppose that for some $j \in \mathcal{N}$, there exist $\theta_j^k, \theta_j^\ell \in \Theta_j$ such that $\beta_j(\theta_j^k) \neq \beta_j(\theta_j^\ell)$, and that $\Pr(\theta) > 0$ for all $\theta \in \Theta$. Then, the quadratic funding mechanism is efficient if, and only if, $\gamma = 1/2$.*

Thus, we can conclude that in games of public goods provision and incomplete information, QF is efficient under very restricted circumstances. Cases where individual contributions are aggregated by the QF mechanism so as to result in efficiency only occur in exceptional situations, as in the example mentioned above. Therefore, in general, the QF mechanism in a framework with incomplete information is not efficient, a finding that stands in contrast with what Buterin et al. (2019) conjectured.

4 Measuring the deadweight loss

Due to the generic inefficiency of the equilibrium that QF produces when the game has incomplete information, in this section we analyze the size of such inefficiency as well as its sensitivity to variations in fundamental parameters of the model. We are going to define some measures of the size of the inefficiency, and analyze their behavior when either information or risk aversion changes or when the size of the population increases.

Before defining the measures of inefficiency, let us introduce the notion of "*second-best*" level of the public good in a game $\mathcal{G} = \{(v_i)_{i\in\mathcal{N}}, \Phi\}$.

Definition 4.1. *We say that $F^{EA} \geq 0$ is an **ex-ante efficient provision** for $\mathcal{G}$ if*

$$F^{EA} = \arg\max_{F\geq 0} E\left[\left(\sum_{i=1}^{N} V_i\left(F;\theta_i\right)\right) - F\right].$$

That is, if it maximizes the expected total social welfare before the individual types are known. From the first order conditions it is characterized by $\sum_{i=1}^{n} E\left[v_i'(F^{EA};\theta_i)\right] \leq 1$, with equality whenever $F^{EA} > 0$.

Notice the difference between the ex-ante and the ex-post efficient provision given in Definition 3.3. The ex-ante efficient provision represents the optimal level of public good provision when the probability distribution of types is used to measure the welfare of the society. Consequently, this provision level is the best that the central planner could choose without using any mechanism to acquire information about the types.

Having defined both ex-ante and ex-post optimal provision levels of the public good, we can define our measures for the size of inefficiency.

Definition 4.2. *The **absolute deadweight loss** for the contribution-based quadratic funding mechanism in the game $\mathcal{G}$ is*

$$\Delta W^A := E\left[W(F^e(\theta)) - W(F^*(\theta))\right].$$

Definition 4.3. *The **relative deadweight loss** for the contribution-based quadratic funding mechanism in the game $\mathcal{G}$ is*

$$\Delta W^R := \frac{E\left[W(F^e(\theta)) - W(F^*(\theta))\right]}{E\left[W(F^e(\theta)) - W(F^{EA})\right]}.$$

In words, the absolute deadweight loss is the expected loss in monetary terms of using the QF equilibrium to fund the public good instead of using the efficient level of the public good for each profile of types that individuals may have. On the other hand, the relative deadweight loss measures the ratio between the absolute deadweight loss and the deadweight loss of providing the ex-ante efficient level of the public good rather than the ex-post efficient level. When this measure is lower (greater) than one, using QF is better (worse) than providing the ex-ante optimal level of the public good.

These measures were inspired by others in the literature (Vives (2002) and Rustichini et al. (1994)); however, to the best of our knowledge were not used in frameworks like the one presented here.

In the next two subsections we will analyze the response of the inefficiency measures proposed above to changes in either the level of uncertainty contained in the incomplete information of the game or in the number of participants in the society.

4.1 Changes in the level of uncertainty

To measure the changes in welfare resulting from changes in the level of uncertainty of the model, we consider the next simple two-player setting. There are two individuals ($n = 2$), $|\Theta_1| = 2$, and $|\Theta_2| = 1$. Let $\alpha := \Pr(\theta_1 = \theta_1^1)$ be the probability of individual $i = 1$ has the type θ_1^1. The utility functions for the public good of these individuals, given their possible types, are

$$v_1(F;\theta_1^1) = \frac{F^{1-\gamma}}{1-\gamma},$$
$$v_1(F;\theta_1^2) = \beta_1\frac{F^{1-\gamma}}{1-\gamma},$$
$$v_2(F;\theta_2^1) = \beta_2\frac{F^{1-\gamma}}{1-\gamma},$$

where $\beta_1, \beta_2 \in [0.1, 50]$. We now solve the individual problems.

The first individual solves, for each of her types,

$$\max_{c_1\geq 0} \tilde{\beta}_1\frac{\left(c_1^{1/2} + c_2(\theta_2^1)^{1/2}\right)^{2(1-\gamma)}}{1-\gamma} - c_1,$$

where $\tilde{\beta}_1$ is either 1 or β_1 if θ_1^i is either θ_1^1 or θ_1^2, respectively. Solving for each type, it results

$$\left(c_1(\theta_1^1)\right)^{1/2} = \left[\left(c_1(\theta_1^1)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\gamma}. \tag{4}$$

$$\left(c_1(\theta_1^2)\right)^{1/2} = \beta_1\left[\left(c_1(\theta_1^2)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\gamma}. \tag{5}$$

Analogously, for individual 2, we have

$$\max_{c_2\geq 0} \alpha\left(\beta_2\frac{\left(c_1^{1/2}(\theta_1^1) + c_2^{1/2}\right)^{2(1-\gamma)}}{1-\gamma}\right) + (1-\alpha)\left(\beta_2\frac{\left(c_1^{1/2}(\theta_1^2) + c_2^{1/2}\right)^{1-\gamma}}{1-\gamma}\right) - c_2$$

$$\begin{aligned}\Rightarrow \left(c_2(\theta_2^1)\right)^{1/2} = {} & \alpha\beta_2\left[\left(c_1(\theta_1^1)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\gamma} \\ & + (1-\alpha)\beta_2\left[\left(c_1(\theta_1^2)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\gamma}.\end{aligned} \tag{6}$$

Solving the equations (4), (5) and (6) numerically, we obtain the QF equilibrium for this game.

The ex-post efficient level of the public good is the solution of:

$$\max_{F\geq 0}(\tilde{\beta}_1+\beta_2)\frac{F^{1-\gamma}}{1-\gamma}-F,$$

which implies that

$$F^e(\theta_1^1,\theta_2^1)=(1+\beta_2)^{1/\gamma}. \tag{7}$$

$$F^e(\theta_1^2,\theta_2^1)=(\beta_1+\beta_2)^{1/\gamma}. \tag{8}$$

Lastly, the ex-ante optimal provision level for the public good solves

$$\max_{F\geq 0}\alpha\left[\frac{F^{1-\gamma}}{1-\gamma}\right]+(1-\alpha)\left[\beta_1\frac{F^{1-\gamma}}{1-\gamma}\right]+\beta_2\frac{F^{1-\gamma}}{1-\gamma}-F,$$

from which it results that

$$F^{EA}=(\alpha+(1-\alpha)\beta_1+\beta_2)^{1/\gamma}. \tag{9}$$

For the first group of numerical illustrations, let us fix $\beta_1=2$, $\beta_2=1$ and consider two alternative values for the risk aversion coefficient, $\gamma=1$ and $\gamma=1/4$. Those values of γ are chosen because $\gamma=1/2$ is a threshold value, from which individuals have different responses to increases in the other participants' contributions, as we stated in proposition 2.5 for the complete information setting.

Let us start varying α to capture the effect of the degree of uncertainty on welfare. In Figure 1, the resulting inverted "U" shape has an intuitive explanation: the closer the game is to complete information ($\alpha=0$ or $\alpha=1$) the lower the deadweight loss is, and it is highest at about halfway between these values ($\alpha\approx 0.48$ and $\alpha\approx 0.53$) depending on the value of γ.

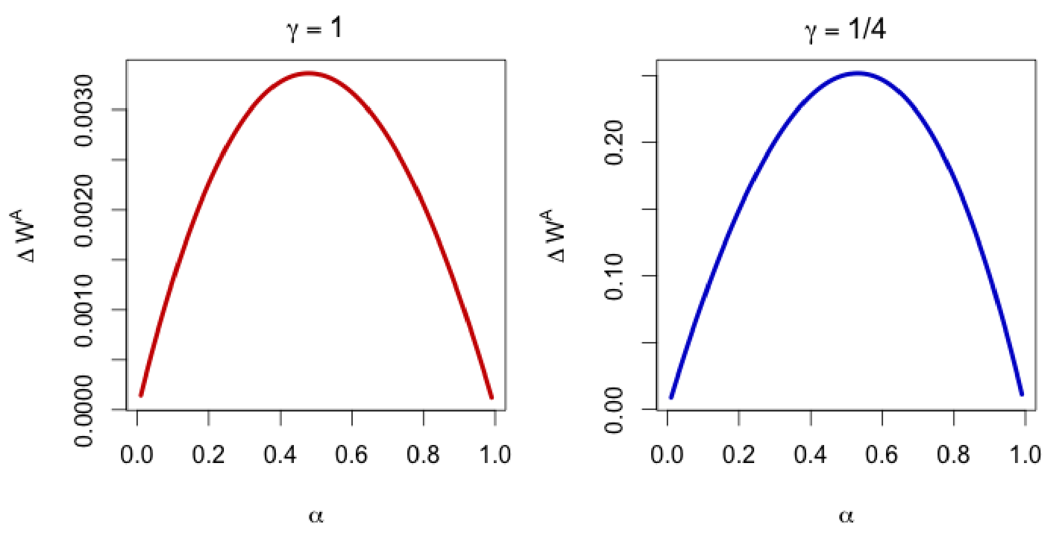

Figure 1: Absolute deadweight loss as a function of $\alpha\in[0,1]$.

Now, let us analyze the relative deadweight loss when α varies in $(0,1)$. Figure 2 shows the monotonic response to increases in the probability of individual 1 having lower valuation for the public good. Notice that ΔW^R is not defined for $\alpha = 0$ or $\alpha = 1$, since the ex-ante optimal provision is efficient in those cases. When $\gamma = 1$, increases in the probability of individual 1 being of the lower type makes QF less inefficient than the the ex-ante optimal provision level. This is compatible with the stabilizing behavior of the best response functions in this case, reported in proposition 2.5. When $\gamma = 1/4$ the opposite behavior is observed augmenting the inefficiency with respect to that of the ex-ante provision level. In both cases, QF is considerably more efficient than the ex-ante optimal provision, as the relative deadweight loss ranges between 0.05 and 0.11.

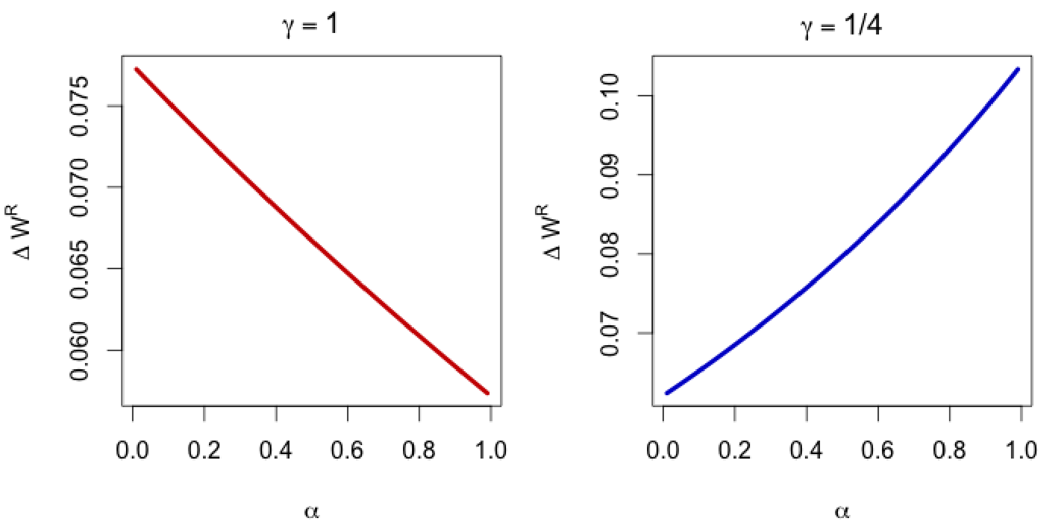

Figure 2: Relative deadweight loss as a function of $\alpha \in (0, 1)$.

Next, we describe the response of the changes in welfare to variation in the *intensity* of the incomplete information shock. Namely, fixing the probability of being of type θ_1^1, $\alpha = 1/2$ we vary the value of the shock β_1. In Figure 3 we can observe that for both values of γ, the absolute deadweight loss increases as the game moves away from the case of complete information ($\beta_1 = 1$). When $\gamma = 1 > 1/2$, an increase in β_1 generates an increase in individual 1's contributions (for both types) and a decrease in the individual 2's contribution. As β_1 grows arbitrarily large, the second term on the right-hand side of 6 goes to zero. Then, individual 2's contribution converges to a positive value, that explains the concave shape of the function for $\beta_1 > 1$. For $\beta_1 < 1$, the same logic implies that individual 2's contribution grows at increasing rates, and so we observe a convex behavior. When $\gamma = 1/4 < 1/2$, an analogous reasoning explains the convex shape observed in the figure.

Figure 4 presents the plots for the relative deadweight loss when varying β_1. We only consider values $\beta_1 > 1$, since ΔW^R is not defined when $\beta_1 = 1$. In both cases, ΔW^R converges to zero as β_1 goes to infinity, indicating that the inefficiency of the ex-ante optimal provision becomes even greater than that of the QF mechanism when the intensity of the incomplete information grows. It shows that the ex-ante level of the public good does not bring as much information as the QF mechanism for the provision of the public good, thus the later is less and less inefficient than the former as the intensity of the incomplete information increases.

To finalize this subsection we analyze the impact of variations in the risk aversion param-

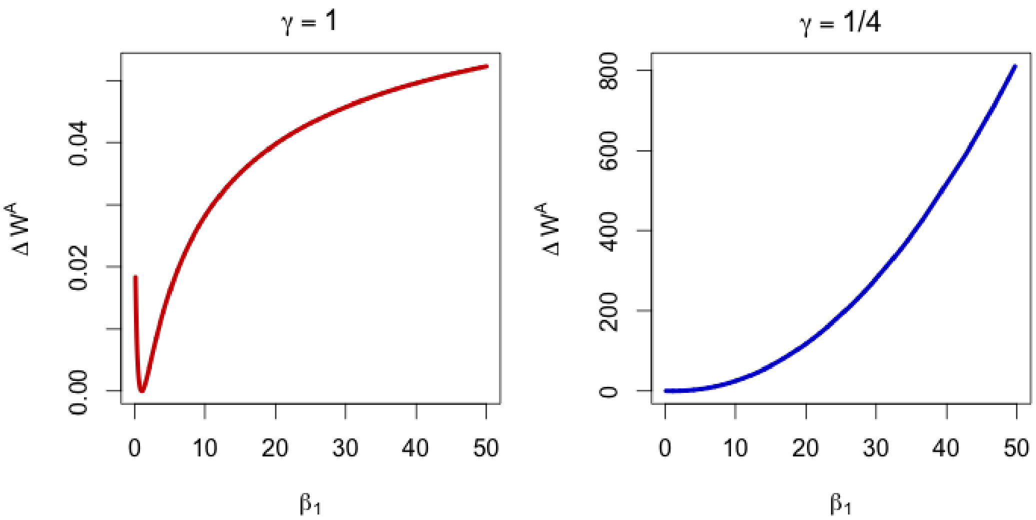

Figure 3: Absolute deadweight loss as a function of $\beta_1 \in [0.1, 50]$.

eter on the relative deadweight loss. In Figure 5 we fix $\alpha = 1/2$ and $\beta_1 = 2$, and vary the value of the relative risk aversion coefficient in the interval [0.1, 5]. This interval is compatible with values for this parameter found in empirical works (Huang et al. (2008) and Gandelman and Hernández-Murillo (2014)). We can observe that it approaches efficiency as γ goes to 1/2, just as proposition 3.5 asserted. When γ moves away from this value, the inefficiency of QF augments more than that of the ex-ante level; however, the relative inefficiency remains below the unity, meaning that QF is less inefficient than the second best ex-ante optimal level of funding for the public good.

4.2 Changes in the population size

To analyze the welfare changes as a response to population size increases, we consider an example where each individual may have one of two types with the same utility functions. Specifically, for $i \in \mathcal{N}$, let $|\Theta_i| = 2$, let $\Pr(\theta_i = \theta_i^1) = \Pr(\theta_i = \theta_i^1 \mid \theta_j) = 1/2$ for all $i, j \in \mathcal{N}$ and $i \neq j$, and let their utility functions of consuming the public good for each type be given by

$$v_i(F; \theta_i^1) = \frac{F^{1-\gamma}}{1-\gamma}$$
$$v_i(F; \theta_i^2) = 2\frac{F^{1-\gamma}}{1-\gamma}.$$

Given the above setup, we have that the probability of $0 \le k \le n$ individuals being of type 1 follows a Bernoulli distribution. As individuals are symmetric, their contributions to the public good are identical whenever they have the same type, so let us use the notation $x^1 := c_i(\theta_i^1)$ and $x^2 := c_i(\theta_i^2)$ to refer to these contributions. The first order conditions for the problem of an individual with type 1 can be written as

$$(x^1)^{1/2} = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \left[(k+1)(x^1)^{1/2} + (n-1-k)(x^2)^{1/2}\right]^{1-2\gamma}. \tag{10}$$

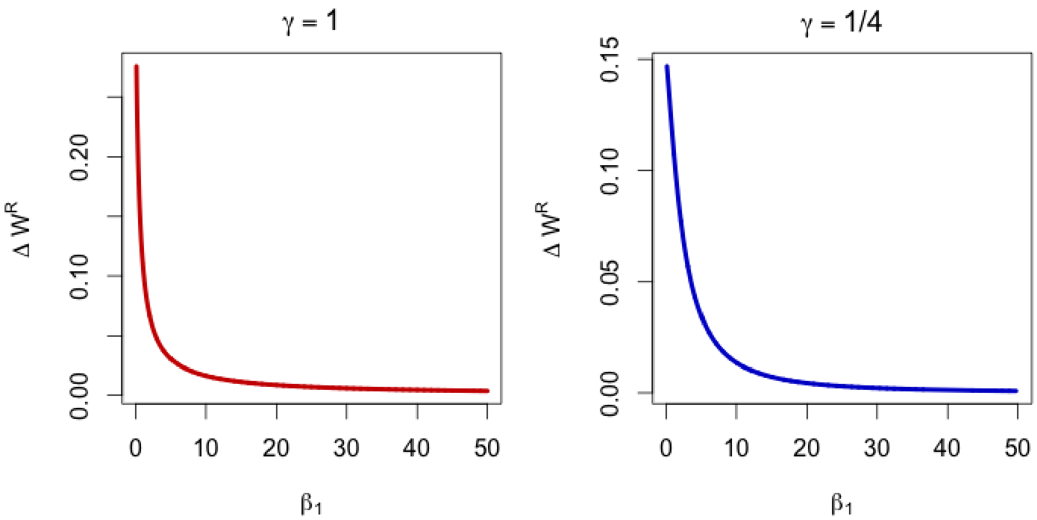

Figure 4: Relative deadweight loss as a function of $\beta_1 \in [0.1, 50]$.

Respectively, for an individual with type 2:

$$(x^2)^{1/2} = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} 2 \left[(k(x^1)^{1/2} + (n-k)(x^2)^{1/2} \right]^{1-2\gamma}. \tag{11}$$

Solving the equations (10) and (11) numerically, we obtain the QF equilibrium for this game.

The ex-post efficient provision level only depends on the number of individuals with a certain type. For a state of the world $\theta \in \Theta$ where the number of individuals with type 1 is $0 \leq k \leq n$, the efficient provision solves

$$\max_{F \geq 0} k \frac{F^{1-\gamma}}{1-\gamma} + 2(n-k) \frac{F^{1-\gamma}}{1-\gamma} - F,$$

whose first order conditions imply that

$$F^e(\theta) = (2n-k)^{1/\gamma}. \tag{12}$$

And finally, the ex-ante optimal provision level must solve

$$\max_{F \geq 0} \left[\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k + 2(n-k)) \frac{F^{1-\gamma}}{1-\gamma} \right] - F,$$

whose explicit solution is

$$F^{EA} = \left[\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (2n-k) \right]^{1/\gamma}. \tag{13}$$

With the analytic forms found above we are going to check two possible properties that the deadweight loss could have: if the inefficiency converges to zero as the population size

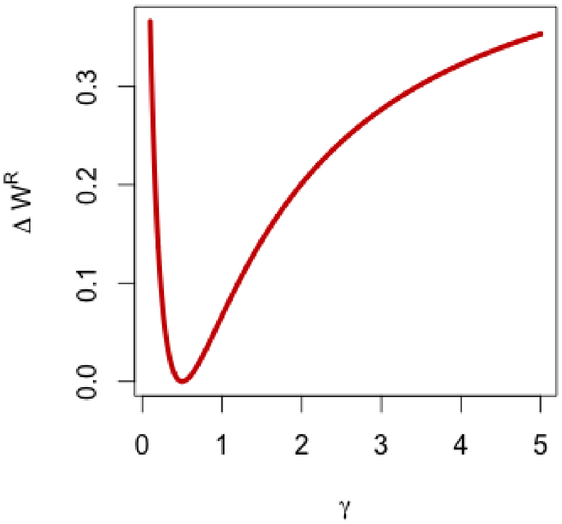

Figure 5: Relative deadweight loss as a function of $\gamma \in [0.1, 5]$.

increases and/or if at least the per capita inefficiency goes to zero. Some works in the literature assessed such questions (Lalley and Weyl (2019) and Rustichini et al. (1994)). We also analyze the asymptotic behavior of the relative deadweight loss as the number of participants goes to infinity.

Firstly, let us think about the two evident effects that an increase in the population size brings to the equilibrium. One of them is that larger population increases, *ceteris paribus*, the number of contributions to the public good, which in turn would augment the variance of the provision level. This effect intensifies the problem generated by incomplete information, resulting in an increase in deadweight loss. On the other hand, a second effect takes place, which is dependent on γ. When n increases, as the extra individuals make positive contributions, this would *ceteris paribus* raise the level of funding for the public good. However, proposition 2.5 asserted (at least in the complete information case) that different values of γ result in different responses of individual contributions to changes in aggregate provision. Namely, if $\gamma > 1/2$, individuals reduce their contributions, converging to zero as n approaches infinity. This promotes a reduction in the dispersion of contributions, and thus lowers the problem caused by incomplete information. The opposite occurs when $\gamma < 1/2$, so the variance of contribution increases, contributing to an increase in inefficiency. From now on, we will refer these effects as *contributor quantity effect* and *contribution dispersion effect*, respectively.

Figure 6 shows the absolute deadweight loss for $\gamma = 1$. Here, as expected, the contribution dispersion effect does lower the rate of growth of inefficiency. The intensity of the effect is large enough to stabilize the deadweight loss, however it does not converge to zero. Hence, we conclude that QF is not asymptotically efficient. On the other hand, we can see that the deadweight loss per capita does converge to zero. It is worth noting that there is a peak at $n = 4$, showing that the contributor quantity effect is strong enough for $2 \leq n \leq 4$ to make the deadweight loss per capita increase with population.

Now, in Figure 7 we have the case $\gamma = 1/4$. The combination of a positive contributor

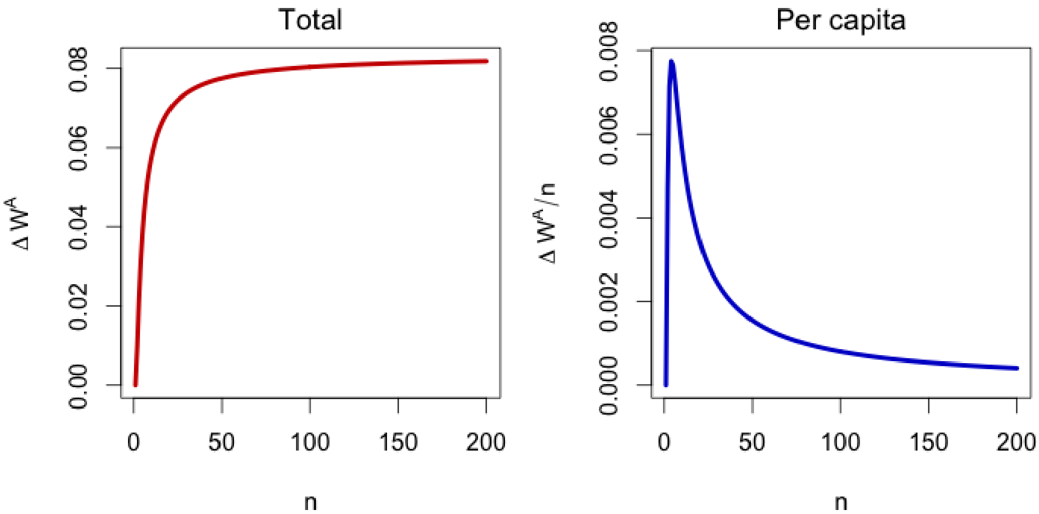

Figure 6: Absolute deadweight loss as a function of $1 \leq n \leq 200$, for $\gamma = 1$.

quantity effect and a positive contribution dispersion effect makes the total absolute deadweight loss grow at increasing rates, reaching the largest order of magnitude of all figures presented until now. The contribution dispersion effect is so strong that even the absolute deadweight loss per capita is a convex function of n.

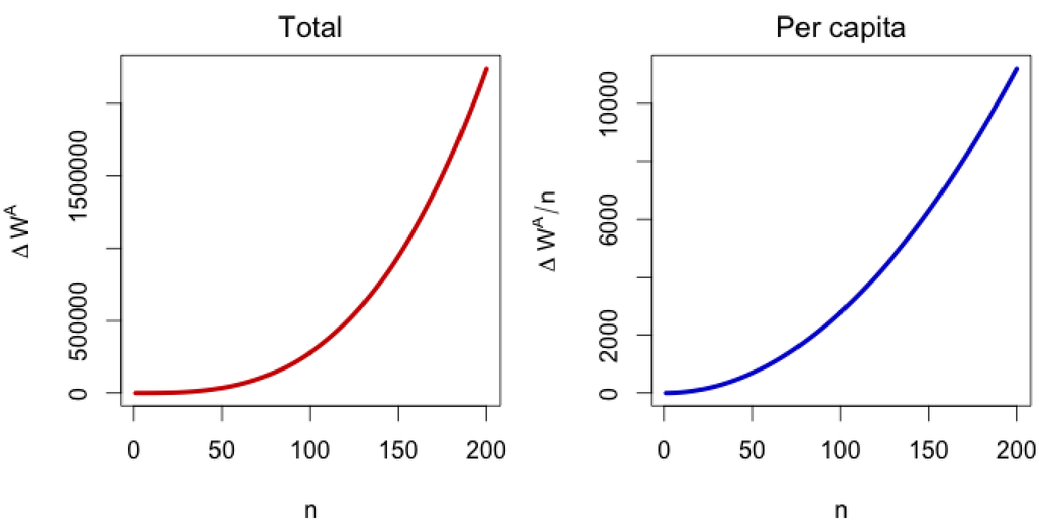

Figure 7: Absolute deadweight loss as a function of $1 \leq n \leq 200$, for $\gamma = 1/4$.

Finally, we show the relative deadweight loss in Figure 8. Notice that all these graphs exhibit the same shape, growing at decreasing rates and (apparently) converging to a positive value. This highlights the advantage of QF in letting individuals choose their own contributions using their private information; however, a larger population diminishes the relative importance of the information possessed by each individual, thus lowering the comparative efficiency of this mechanism.

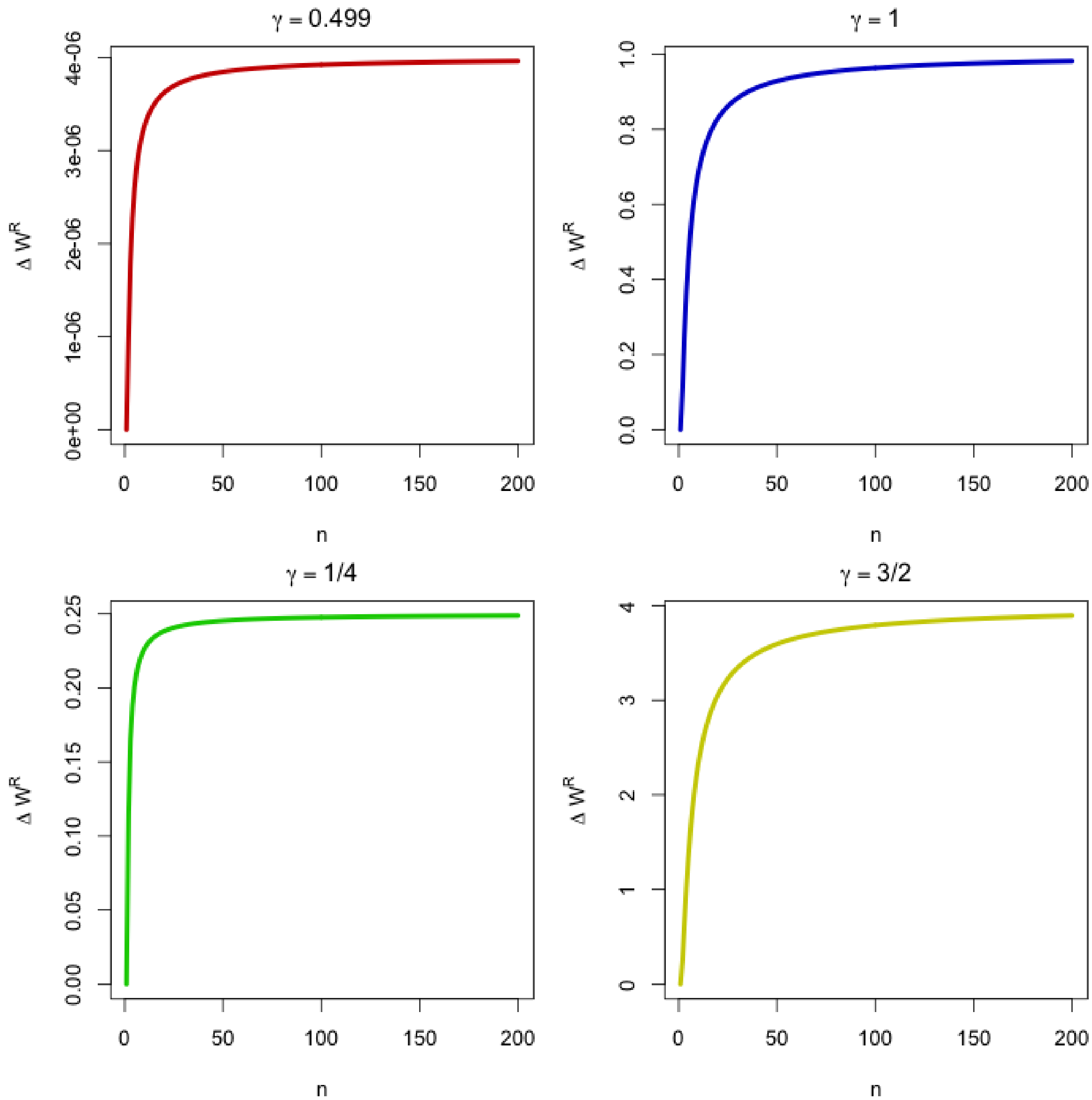

Figure 8: Relative deadweight loss as a function of $1 \leq n \leq 200$.

Let us comment each case in that figure. For $\gamma = 0.499$ (slightly lower than $1/2$) we already argued that the contribution dispersion effect is much lower than the contributor quantity effect, making the absolute deadweight loss jointly increase with the population size. However, the inefficiency is much lower than the ex-ante inefficiency, thus the relative inefficiency is very low. In the case of $\gamma = 1$ the inefficiency measure converges to 1 when n goes to infinity, indicating that QF is asymptotically equivalent to the ex-ante provision in terms of efficiency. When $\gamma = 1/4$, despite the absolute deadweight loss becoming considerably high for large populations, the ex-ante optimal provision has an even higher expected inefficiency for larger populations. Lastly, if $\gamma = 3/2$, we can see that QF becomes more inefficient than using the ex-ante optimal provision.

5 Conclusions

Mechanisms to attain the efficiency of decentralized public good provision are widely studied and discussed in the literature. The difficulties aroused by the possible collusion formations or inefficiencies of the majority rule bring a challenge for theoretical and applied modeling of some of those mechanisms. In this sense, the quadratic funding mechanism for providing a public good appears as a solution for reaching the Pareto optimality in a decentralized way. Its simplicity and optimality when the efficient amount is strictly positive make it a promising scheme for financing public goods whenever the individuals have complete information regarding their peers' preferences.

In this work, we analyze the extent to which efficiency is maintained when there is incomplete information in the model. The conclusions are that only under very restrictive conditions on the utility functions of the participants is it possible to reach efficiency. We provide several necessary and sufficient conditions to guarantee the efficiency of the equilibrium. In particular, we show that, for the class of CRRA utility functions representing the participants' preferences for the public good, the efficiency results if and only if the risk aversion coefficient is equal to $\frac{1}{2}$.

We also propose two measures for the inefficiency size (the deadweight loss) for this incomplete information game: the equilibrium's absolute and relative deadweight loss. In absolute (monetary equivalent) terms, the first compares the welfare loss of using the private provision rather than the ex-post optimal provision. The other measure is the ratio between the absolute deadweight loss and the welfare loss of the ex-ante (second-best) optimal provision. That second-best allocation is the central planner's provision given that she only knows the distributions of the individuals' types. The ratio compares the inefficiency of the private provision with that of the second-best alternative. We provide several analyses of how those measures vary when parameters of the incomplete information or the number of participants change.

We believe that the results presented here may contribute to the better applying of the QF mechanism, given that in many situations, the contributors do not have complete information regarding the benefits that the public good may bring to the peers. Additionally, estimating changes to a measure of relative deadweight loss similar to the one develped here when the fundamentals of the model vary can be valuable tools for Public Economics.

Acknowledgements

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Appendix

Proof. (Proposition 2.1) Let the function $v : \mathbb{R}_+ \to \mathbb{R}$ be defined as $v(F) = \sum_{i=1}^n v_i(F)$. With the assumptions given, we have that $v \in C^1$ is strictly concave, and that $\lim_{F\to\infty} v'(F) = 0$. We then have two cases: $v'(0) \leq 1$ or $v'(0) > 1$. In the first case, it follows from 2.4 that $F = 0$ is optimal. Furthermore, since v is a strictly concave function, we have that $v'(F) < 1$ for all $F > 0$ and thus, again from 2.4, there can be no socially optimal provision $F > 0$. Thus, $F^e = 0$ is the unique efficient provision.

Now suppose that $v'(0) > 1$. It follows that $F = 0$ is not optimal. As $\lim_{F\to\infty} v'(F) = 0$, we have that there exists $A \in \mathbb{R}$ such that $v'(A) < 1$. Thus, since v' is continuous, $v'(0) > 1$ and $v'(A) < 1$, it follows that there exists $0 < B < A$ such that $v'(B) = 1$. Additionally, since v' is strictly decreasing, we have that $v'(F) \neq 1$ for all $F \neq B$. Therefore, $F^e = B$ is the unique efficient provision.

We then have that in both cases there is a unique efficient provision $F^e \geq 0$, as desired. □

Proof. (Proposition 2.2) The first order condition for the individual i having and interior solution is:

$$v_i'(\Phi(\mathbf{c}))\big(\frac{1}{\rho}\big)\big[\sum_{i=1}^n c_i^\rho\big]^{(1/\rho)-1}(\rho c_i^{\rho-1}) = 1 \Rightarrow v_i'(\Phi(\mathbf{c})) = \frac{c_i^{1-\rho}}{\big[\sum_{i=1}^n c_i^\rho\big]^{(1/\rho)-1}}.$$

Summing up on i and arranging:

$$\sum_{i=1}^n v_i'(\Phi(\mathbf{c})) = \frac{\sum_{i=1}^n c_i^{1-\rho}}{\big[\sum_{i=1}^n c_i^\rho\big]^{(1/\rho)-1}} = \left[\frac{\big[\sum_{i=1}^n c_i^{1-\rho}\big]^{1/(1-\rho)}}{\big[\sum_{i=1}^n c_i^\rho\big]^{1/\rho}}\right]^{1-\rho} \equiv E. \tag{14}$$

It is clear that if $\rho = 1/2$, then $E = 1$; therefore $\Phi(\mathbf{c}) = F^e$ and (*iii*) results.

Now, let us define the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(\rho) = [\sum_{i=1}^n c_i^\rho]^{1/\rho}$. We are going to prove that f is a strictly increasing function. Taking the logarithm and the derivative:

$$\begin{aligned}\frac{f'(\rho)}{f(\rho)} &= \frac{\sum_{i=1}^n c_i^\rho \ln(c_i)}{\rho\sum_{i=1}^n c_i^\rho} - \frac{\ln(\sum_{i=1}^n c_i^\rho)}{\rho^2} \\ &= \frac{1}{\rho^2\sum_{i=1}^n c_i^\rho}\left\{\rho\sum_{i=1}^n c_i^\rho \ln(c_i) - \left(\sum_{i=1}^n c_i^\rho\right)\ln\left(\sum_{i=1}^n c_i^\rho\right)\right\}\end{aligned}$$

$$\Rightarrow f'(\rho) = \frac{f(\rho)}{\rho^2 \sum_{i=1}^n c_i^\rho} \left\{ \sum_{i=1}^n c_i^\rho \left[\ln(c_i^\rho) - \ln\left(\sum_{i=1}^n c_i^\rho \right) \right] \right\}.$$

Since the term in brackets is always negative, $f'(\rho) < 0$ and f is a strictly decreasing function. Finally, since $E = \left[\frac{f(1-\rho)}{f(\rho)}\right]$, $v = \sum_{i=1}^n v_i$ is a strictly concave function and $\rho < 1$, from (14) we can conclude:

$$\Phi(\mathbf{c}) < F^e \Leftrightarrow E > 1 \Leftrightarrow f(1-\rho) > f(\rho) \Leftrightarrow 1-\rho < \rho \Leftrightarrow \rho > \frac{1}{2}.$$

Analogously, $\Phi(\mathbf{c}) > F^e \Leftrightarrow \rho < \frac{1}{2}$. Thus (i) and (ii) are proved.

□

Proof. (Proposition 2.3) By proposition 2.1, we know that the hypotheses adopted here guarantee that there is a unique socially optimal provision $F^e \geq 0$. There are two cases: $F^e = 0$ or $F^e > 0$. First, suppose we have $F^e = 0$. We are going to show that $\mathbf{0}$ is an equilibrium for Φ^{QF}. Consider the problem faced by some individual $i \in \mathcal{N}$ when all other individuals are contributing zero to the mechanism:

$$\max_{c_i \geq 0} v_i \left(\left[c_i^{1/2} + 0 \right]^2 \right) - c_i.$$

Which can be written as

$$\max_{c_i \geq 0} v_i\left(c_i\right) - c_i.$$

Thus, the first order condition for the individual i is that $v_i'(c_i) \leq 1$, with equality holding when $c_i > 0$. But note that, since $F^e = 0$, we have from definition 2.4 that $\sum_{j=1}^n v_j'(0) \leq 1$. In particular, since v_j is increasing for all $j \in \mathcal{N}$, this implies that $v_i'(0) \leq 1$. Thus, $c_i = 0$ satisfies the first order condition for i. But since the choice of i was arbitrary, we have that $\mathbf{0}$ is an equilibrium for Φ^{QF}.

Now the other case, suppose $F^e > 0$. For all $i \in \mathcal{N}$, let

$$c_i = \left(v_i'(F^e) \cdot (F^e)^{1/2} \right)^2. \tag{15}$$

It is easy to check that $F^e = \Phi^{QF}(\mathbf{c}) = [\sum_{i=1}^n c_i^{1/2}]^2$. Rearranging, we obtain

$$v_i'(F^e) = \frac{(c_i)^{1/2}}{(F^e)^{1/2}},$$

which is precisely the first order condition for the individual i's optimization problem in the QF mechanism. We can thus conclude that the vector $\mathbf{c} = (c_1, \cdots, c_n)$ as defined by (15) is an equilibrium of QF and its provision is optimal.

Thus, in all cases, there exists an equilibrium allocation $\mathbf{c}^*$ such that $\Phi^{QF}(\mathbf{c}^*) = F^e$, as we wanted to show.

□

Proof. (Proposition 3.1) Let $v : \mathbb{R}_+ \times \Theta \to \mathbb{R}$ be defined by $v(F;\theta) = \sum_{i=1}^{n} v_i(F;\theta_i)$. By a straightforward argument similar to that presented in the proof of 2.1 we can prove that for each $\theta \in \Theta$, there is a unique efficient funding $F^e(\theta) \geq 0$. In this way, we construct a unique function $F^e : \Theta \to \mathbb{R}_+$ that maps each profile of types to its efficient funding. □

Proof. (Lemma 3.1) Without loss of generality, let us consider $i = 1$. Let $\theta_1 \in \Theta_1$ be the type of this individual, and suppose that there exist $a, b \in \mathbb{R}_+$, $a < b$, best-responses to $\mathbf{c}_{-i}$. Let $\varepsilon = b^{1/2} - a^{1/2} > 0$. From the first order conditions, we have

$$\begin{aligned} a^{1/2} &= E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \cdot \left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right] \middle| \theta_i \right] \\ &= a^{1/2} E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \middle| \theta_i \right] \\ &+ E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \cdot \left[\sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right] \middle| \theta_i \right]. \end{aligned}$$

Notice that the term in the third line above is nonnegative; thus, the equality above is satisfied only if $E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \middle| \theta_i \right] \leq 1$. Thus, since $b > a$ and v_1' is strictly decreasing, we have that

$$\begin{aligned} &E\left[v_1' \left(\left[b^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \cdot \left[\sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right] \middle| \theta_i \right] \\ &< E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \cdot \left[\sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right] \middle| \theta_i \right], \end{aligned}$$

and

$$\begin{aligned} &b^{1/2} E\left[v_1' \left(\left[b^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \middle| \theta_i \right] \\ &< a^{1/2} E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \middle| \theta_i \right] + \varepsilon. \end{aligned}$$

Thus, adding the inequalities and using the fact that a and b satisfy the first order conditions, it follows that $a^{1/2} + \varepsilon > b^{1/2}$. But this contradicts the definition of ε. This contradiction completes the proof. □

Proof. (Lemma 3.2) Under the given hypotheses, we can then define, for each $i \in \mathcal{N}$, a function $f_i : \Theta_i \to \mathbb{R}_+$ mapping each θ_i to $f(\theta_i) > 0$ such that $v_i'(f(\theta_i);\theta_i) < 1/n$. Now, letting $A := \max\{f_i(\theta_i); i \in \mathcal{N}, \theta_i \in \Theta_i\}$, by the hypothesis of strict concavity of v_i it follows that $v_i'(A;\theta_i) < 1/n$, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$. Now, suppose that, for some $j \in \mathcal{N}$, we have that $c_i(\theta_i) \in [0, A]$ for all $i \neq j$ and all $\theta_i \in \Theta_i$, and the best-response of j with type $\theta_j \in \Theta_j$ is $k > A$. Since $k > 0$, the first order conditions for individual j's problem imply that

$$\begin{aligned} k^{1/2} &= E\left[v_j'\left(\left[k^{1/2} + \sum_{i \neq j} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_j \right) \cdot \left[k^{1/2} + \sum_{i \neq j} (c_i(\theta_i))^{1/2} \right] \middle| \theta_j \right] \\ &< \frac{1}{n} E\left[k^{1/2} + \sum_{i \neq j} (c_i(\theta_i))^{1/2} \middle| \theta_j \right] \\ &< \frac{1}{n} n \cdot k^{1/2}, \end{aligned}$$

where the first inequality follows from $v_j'(A; \theta_j) < 1/n$ and $v_j(\cdot; \theta_j)$ being strictly concave, and the second inequality follows from $c_i(\theta_i) \in [0, A]$ for all $i \neq j$. It results that $k < k$; this contradiction completes the proof. □

Proof. (Proposition 3.2) For each individual $i \in \mathcal{N}$, let $|\Theta_i| = L_i$. We can restrict the domain and range of the best-response functions to the interval $[0, A]$, using $A > 0$ given in lemma 3.2. Thus, the problem of individual i with type θ_i is

$$\max_{c_i \in [0, A]} E\left[v_i\left(\Phi^{QF}(c_i, \mathbf{c}_{-i}(\theta_{-i})); \theta_i \right) \middle| \theta_i \right] - c_i.$$

Notice that the function that is being maximized is continuous and the feasibility correspondence is continuous and compact valued (it is the constant interval $[0, A]$). Thus, by the Theorem of the Maximum (Berge (1963), ch. 6), we have that the best response correspondence for i is not empty and is upper hemicontinuous. Additionally, by lemma 3.1, we can conclude that it is a continuous function. Hence, the whole game best-response function $BR : [0, A]^{\sum_{i=1}^n L_i} \to [0, A]^{\sum_{i=1}^n L_i}$ is also continuous. Since $[0, A]^{\sum_{i=1}^n L_i}$ is a compact and non-empty set, the Brouwer fixed point theorem (Milnor (1965)) allows us to conclude that BR has a fixed point which is clearly an equilibrium for QF. □

Proof. (Proposition 3.3) First, suppose that $F^e(\theta) = 0$ for all $\theta \in \Theta$. Let us prove that $(\theta) = 0$ for all θ is an equilibrium for QF. Suppose that, for some $i \in \mathcal{N}$ we have that all other individuals are playing the strategy profile $\mathbf{c}_{-i}(\theta_{-i}) = 0$. The problem faced by the individual i with some type $\theta_i \in \Theta_i$ is given by

$$\max_{c_i \geq 0} E\left[v_i\left(\Phi^{QF}(c_i, \mathbf{c}_{-i}(\theta_{-i})); \theta_i \right) \middle| \theta_i \right] - c_i.$$

Substituting $\mathbf{c}_{-i}(\theta_{-i}) = 0$ and the definition of QF, the problem above becomes

$$\max_{c_i \geq 0} v_i\left(c_i; \theta_i \right) - c_i,$$

let $c_i(\theta_i)$ be the solution, then the first order conditions are

$$v_i'(c_i(\theta_i); \theta_i) \leq 1,$$

with equality holding if $c_i(\theta_i) > 0$. Since $F^e(\theta) = 0$ for all $\theta \in \Theta$, it implies that $\sum_{j=1}^n v_j'(0; \theta_j) \leq 1$. It follows that $v_i'(0; \theta_i) \leq 1$. Thus, $c_i(\theta_i) = 0$ satisfies the first order conditions for i, as we wanted to show.

Let us prove the reciprocal using a contradiction argument. The hypothesis asserts that there exist $\theta' \in \Theta$ for which $F^e(\theta') = 0$ and that QF is efficient. Then, suppose that there is a $\theta'' \in \Theta$, such that $F^e(\theta'') > 0$. Let $\mathbf{c}$ be an efficient equilibrium for QF. The first order condition for individual i's problem for the type profile θ'' is

$$E\left[v_i'(F^*(\theta)); \theta_i'') \cdot \left(\frac{F^*(\theta)}{c_i^*(\theta_i'')}\right)^{1/2} \middle| \theta_i''\right] \leq 1, \tag{16}$$

which cannot be satisfied for $c_i^*(\theta_i'') = 0$, since $\Pr(\theta'') > 0$, $F^*(\theta'') = F^e(\theta'') > 0$, and $v_i'(F^*(\theta''); \theta_i'') > 0$. Therefore, $c_i^*(\theta_i'') > 0$. Since i is arbitrary, it results $\mathbf{c}^*(\theta'') \gg \mathbf{0}$. Now, let us consider the first order condition of individual i's problem when her type is θ_i',

$$E\left[v_i'(F^*(\theta)); \theta_i') \cdot \left(\frac{F^*(\theta)}{c_i^*(\theta_i')}\right)^{1/2} \middle| \theta_i'\right] \leq 1, \tag{17}$$

with equality holding when $c_i^*(\theta_i') > 0$. Since $\Pr(\theta_i', \theta_{-i}'') > 0$, and $\mathbf{c}_{-i}^*(\theta_{-i}'') \gg \mathbf{0}$, we have that $c_i^*(\theta_i') > 0$, because the numerator of the expected value when $\theta = (\theta_i', \theta_{-i}'')$ is strictly positive. Thus, $c_i^*(\theta_i') > 0$. But then, $\Phi^{QF}(\mathbf{c}(\theta')) > 0 = F^e(\theta')$, which is a contradiction to the efficiency of QF. This completes the proof. □

Proof. (Proposition 3.4) Without loss of generality, let $j = 1$. First, suppose that QF is efficient, that is, there exists an equilibrium strategy profile $\mathbf{c}^*$ such that $\Phi^{QF}(\mathbf{c}^*(\theta)) = F^*(\theta) = F^e(\theta)$. For $\theta \in \Theta$, the optimality of $F^e(\theta) > 0$ implies

$$v_1'(F^e(\theta); \theta_1) + \sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = 1.$$

Thus,

$$\sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = 1 - v_1'(F^e(\theta); \theta_1). \tag{18}$$

Since $j = 1$ has complete information regarding the other individuals' preferences, it follows that

$$v_1'(F^e(\theta); \theta_1) = \frac{(c_1(\theta))^{1/2}}{(F^e(\theta))^{1/2}}. \tag{19}$$

Thus, equations (18) and (19) imply that

$$\sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = \frac{\sum_{i=2}^{n}(c_i(\theta_i^1))^{1/2}}{(F^e(\theta))^{1/2}},$$

and so letting $A = \sum_{i=2}^{n}(c_i(\theta_i^1))^{1/2}$ yields the desired result.

To prove the converse, suppose that there exists a constant $A \in \mathbb{R}$ such that $\sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = A(F^e(\theta))^{-1/2}$. Define the contribution of individual $i \in \mathcal{N}$ with type $\theta_i \in \Theta_i$ by

$$c_i(\theta_i) = E\left(\left[v_i'(F^e(\theta)) \cdot (F^e(\theta))^{1/2} \middle| \theta_i^1\right]\right)^2, \tag{20}$$

which is well-defined since $F^e(\theta)$ exists and is unique. Note that these contributions satisfy the first order conditions if the generated level of the public good is equal to $F^e(\theta)$, so let us prove this equality.

Since each individual $i = 2, \cdots, n$ has only one single type, the conditional expectation in equation (20) is equal to the unconditional expectation. Taking the square root of both sides and taking the sum yields

$$\begin{aligned}
\sum_{i=2}^{n} (c_i(\theta_i^1))^{1/2} &= \sum_{i=2}^{n} E\left[v_i'(F^e(\theta)) \cdot (F^e(\theta))^{1/2}\right] \\
&= E\left[(F^e(\theta))^{1/2} \sum_{i=2}^{n} v_i'(F^e(\theta))\right] \\
&= E\left[(F^e(\theta))^{1/2} \frac{A}{(F^e(\theta))^{1/2}}\right] \\
&= A.
\end{aligned}$$

Thus, substituting the left-hand side on $\sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = A(F^e(\theta))^{-1/2}$ we get

$$\sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = \frac{\sum_{i=2}^{n} (c_i(\theta_i^1))^{1/2}}{(F^e(\theta))^{1/2}} \tag{21}$$

On the other hand, since the individual 1 has complete information, rearranging (20) we get

$$v_1'(F^e(\theta); \theta_1) = \frac{(c_1(\theta_1))^{1/2}}{(F^e(\theta))^{1/2}}. \tag{22}$$

Finally, adding (21) and (22), and using the fact that $\sum_{i=1}^{n} v_i'(F^e(\theta); \theta_i) = 1$ (because $F^e(\theta)$ is the efficient provision), we get that $\Phi^{QF}(\mathbf{c}(\theta)) = F(\theta) = F^e(\theta)$. Thus, it follows that the contributions specified in (20) do indeed satisfy the first order conditions of an equilibrium, and that this equilibrium is efficient, as we wanted to show.

□

Proof. (Proposition 3.5) If $\gamma = 1/2$, then for all $i \in \mathcal{N}$ we have that $v_i'(F; \theta_i) = \beta_i(\theta_i) F^{-1/2}$. For every $i \in \mathcal{N}$, let us define $c_i(\theta_i)$ by

$$c_i(\theta_i) = \left(E\left[v_i'(F^e(\theta); \theta_i) \cdot (F^e(\theta))^{1/2} \,\middle|\, \theta_i\right]\right)^2, \tag{23}$$

which is well-defined since $F^e(\theta)$ exists and is unique. These contributions satisfy the first order conditions of the individual problem. Let us show that $F(\theta) = \Phi(\mathbf{c}(\theta)) = F^e(\theta)$. Substituting $v_i'(F; \theta_i)$ in (23) we get

$$\begin{aligned}
(c_i(\theta_i))^{1/2} &= E\left[\beta_i(\theta_i) \cdot (F^e(\theta))^{-1/2} \cdot (F^e(\theta))^{1/2} \,\middle|\, \theta_i\right] \\
&= \beta_i(\theta_i),
\end{aligned}$$

therefore, we have

$$v_i'(F^e(\theta); \theta_i) = \frac{(c_i(\theta_i))^{1/2}}{(F^e(\theta))^{1/2}}.$$

So, adding this expression for all i,

$$\sum_{i=1}^{n} v_i'(F^e(\theta);\theta_i) = \frac{\sum_{i=1}^{n}(c_i(\theta_i))^{1/2}}{(F^e(\theta))^{1/2}}.$$

Hence, since $F^e(\theta)$ is efficient, we have that the left-hand side of the above equation is equal to one. We then have that $F(\theta) = \left(\sum_{i=1}^{n}(c_i(\theta_i))^{1/2}\right)^2 = F^e(\theta)$, and the equilibrium $\mathbf{c}$ is efficient.

To prove the converse, suppose that QF is efficient and $\gamma \neq 1/2$. By hypothesis, we know that there exists some $j \in \mathcal{N}$ and types $\theta_j^k, \theta_j^\ell \in \Theta_j$ such that $\beta_j(\theta_j^k) \neq \beta_j(\theta_j^\ell)$. Without loss of generality, let $\beta_j(\theta_j^k) > \beta_j(\theta_j^\ell)$. Now, let $\overline{\theta} \in \Theta$ be the state of the world such that $\beta_i(\overline{\theta}_i) \geq \beta_i(\theta_i)$, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$. Let $\mathbf{c}^*$ be the efficient equilibrium. Then, the first order condition for the individual i in the types profile $\overline{\theta}$ imply that

$$(c_i^*(\overline{\theta}_i))^{1/2} = \beta_i(\overline{\theta}_i) \cdot E\left[(F^e(\theta))^{1/2-\gamma} \,\middle|\, \overline{\theta}_i\right]. \tag{24}$$

Let us suppose that $\gamma > 1/2$ (the case $\gamma < 1/2$ is analogous). First, efficiency implies that, for any $\theta \in \Theta$, $\sum_{i=1}^{n} v_i'(F^e(\theta);\theta_i) = \sum_{i=1}^{n} \beta_i(\theta_i) \cdot (F^e(\theta))^{-\gamma} = 1$. Then, as $\beta_i(\overline{\theta}_i) \geq \beta_i(\theta_i)$ for all i and all θ_i, it follows that $\sum_{i=1}^{n} v_i'(F^e(\theta);\overline{\theta}_i) = \sum_{i=1}^{n} \beta_i(\overline{\theta}_i) \cdot (F^e(\theta))^{-\gamma} \geq 1$, for all $\theta \in \Theta$. From strict monotonicity of $v_i'(\cdot;\theta_i)$, we have that $F^e(\overline{\theta}) \geq F^e(\theta)$, for all $\theta \in \Theta$, in particular, $F^e(\overline{\theta}) > F^e(\theta_j^\ell, \overline{\theta}_{-j})$. Thus, as $1/2 - \gamma < 0$, it follows that $(F^e(\overline{\theta}))^{1/2-\gamma} \leq (F^e(\theta))^{1/2-\gamma}$ for all $\theta \in \Theta$ and, in particular, $(F^e(\overline{\theta}))^{1/2-\gamma} < (F^e(\theta_j^\ell, \overline{\theta}_{-j}))^{1/2-\gamma}$. Since $\Pr\left(\theta_j^\ell \,\middle|\, \overline{\theta}_i\right) > 0$ for all $i \neq j$, we then have

$$E\left[F^e(\theta))^{1/2-\gamma} \,\middle|\, \overline{\theta}_i\right] \geq (F^e(\overline{\theta}))^{1/2-\gamma} \tag{25}$$

for all $i \in \mathcal{N}$, with strict inequality when $i \neq j$. By (24) and (25) we have that

$$(c_i^*(\overline{\theta}_i))^{1/2} \geq \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{1/2-\gamma} \tag{26}$$

for all $i \in \mathcal{N}$, with strict inequality when $i \neq j$. Adding (26) across all $i \in \mathcal{N}$ yields

$$\sum_{i=1}^{n}(c_i^*(\overline{\theta}_i))^{1/2} = (F^e(\overline{\theta}))^{1/2} > \sum_{i=1}^{n} \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{1/2-\gamma}. \tag{27}$$

Dividing both sides of the inequality in (27) by $(F^e(\overline{\theta}))^{1/2}$, we get

$$1 > \sum_{i=1}^{n} \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{-\gamma} = \sum_{i=1}^{n} v_i'(F^e(\overline{\theta});\overline{\theta}_i), \tag{28}$$

which is a contradiction to the definition of efficient provision. This completes the proof. □