Department of Economics- FEA/USP

LABOR TIME SHARED IN THE ASSIGNMENT GAME GENERATING NEW COOPERATIVE AND COMPETITIVE STRUCTURES

MARILDA SOTOMAYOR

WORKING PAPER SERIES Nº 2013-02

DEPARTMENT OF ECONOMICS, FEA-USP
WORKING PAPER Nº 2013-02

LABOR TIME SHARED IN THE ASSIGNMENT GAME GENERATING NEW COOPERATIVE AND COMPETITIVE STRUCTURES

Marilda Sotomayor (marildas@usp.br)

Abstract:

Aiming to obtain new characterizations for the concepts of core, cooperative equilibrium and competitive equilibrium, and new correlations among these concepts, we introduce labor time into the assignment game. Two many-to-many matching models are obtained, distinguished by the nature of the agreements - rigid and flexible. An example illustrates that the characteristic function form does not always fully represent the cooperative structure of the two markets. Two different notions of demand correspondence generate distinct sets of competitive equilibrium allocations. The connection between the cooperative structures of both markets and the cooperative and competitive structures of each market is established through five cooperative solution sets proved to be non-empty, distinct and correlated by the set inclusion - one set is a superset of the next: the maximal set is the core; the second one characterizes the cooperative equilibria for the rigid market; the third set characterizes the cooperative equilibria for the flexible market; the other two sets characterize the competitive equilibrium allocations for the two competitive markets.

Keywords: stable allocations, core, competitive equilibrium allocations, feasible deviation

JEL Codes: C78, D78

LABOR TIME SHARED IN THE ASSIGNMENT GAME GENERATING NEW COOPERATIVE AND COMPETITIVE STRUCTURES.

by

MARILDA SOTOMAYOR[1]

Universidade de São Paulo

Department of Economics, Cidade Universitária, Av. Prof. Luciano Gualberto 908,

05508-900, São Paulo, SP, Brazil

e-mail: marildas@usp.br

April, 2012

ABSTRACT

Aiming to obtain new characterizations for the concepts of core, cooperative equilibrium and competitive equilibrium, and new correlations among these concepts, we introduce labor time into the assignment game. Two many-to-many matching models are obtained, distinguished by the nature of the agreements - rigid and flexible. An example illustrates that the characteristic function form does not always fully represent the cooperative structure of the two markets. Two different notions of demand correspondence generate distinct sets of competitive equilibrium allocations. The connection between the cooperative structures of both markets and the cooperative and competitive structures of each market is established through five cooperative solution sets proved to be non-empty, distinct and correlated by the set inclusion - one set is a superset of the next: the maximal set is the core; the second one characterizes the cooperative equilibria for the rigid market; the third set characterizes the cooperative equilibria for the flexible market; the other two sets characterize the competitive equilibrium allocations for the two competitive markets.

Keywords: stable allocations, core, competitive equilibrium allocations, feasible deviation.

JEL numbers: C78, D78

INTRODUCTION

---

[1] This paper was partially supported by CNPq-Brazil and by FIPA/USP. A draft of it was written in 1997 and a preliminary version was written in 2002 and entitled "The multiple partners game with a general quota". Another version was presented at Barcelona-Jocs seminar series, in May, 2009, and entitled: "Correlating new cooperative and competitive solution concepts in the time-sharing assignment game".

The assignment game of Shapley and Shubik (1972)[2] and its many-to-one and many-to-many extensions, have been studied in several papers. One of the peculiarities of these games is that they model environments that can be treated cooperatively and competitively. Thus the two game structures can be examined altogether and compared to each other, which permits that the natural solution concepts of each structure - core, cooperative equilibrium and competitive equilibrium - can be characterized and correlated.

In the many-to-one assignment game introduced by Kelso and Crawford (1982) and studied later by Gul and Stacchetti (1999, 2000), as well as in the many-to-many extensions introduced by Sotomayor (1992, 2002), for example, it is assumed that the agents' payoffs are one-dimensional. These payofs can be interpreted as resulting from negotiations in block. The agents only care about their total payoffs. This assumption causes the core, the set of cooperative equilibrium allocations and the set of competitive equilibrium allocations to coincide.

Sotomayor (1992) also presents another many-to-many matching model, the so called multiple-partners assignment game, which has been widely studied in Sotomayor (1999-a, 2007, 2009) and in Fagebaume *et al* (2010). This model is obtained by introducing quotas into the assignment game and by assuming that agents' utilities are additively separable. This assumption propitiates a multi-dimensional payoff for each individual, one individual payoff corresponding to each trade performed. The negotiations are then pairwise and independent.[3] The agents can be interpreted as being buyers and sellers of indivisible goods and a buyer cannot acquire more than one item from the same seller.

Studies that have been developed for the multiple-partners assignment game have revealed that the independence and multi-dimensionality of the individual payoffs in both sides of the market permit to work with cooperative and competitive structures more theoretically interesting than those provided by markets where the payoffs are one-dimensional or where the multiple-partnerships are restricted to only one of the sides of the market (Sotomayor 1999-a). In fact, unlike these markets, in the multiple-partners assignment game, the core, the set of cooperative equilibrium allocations and

---

[2] See also Roth and Sotomayor (1990) for an overview on the assignment game.

[3] The independence means that an agent's individual payoff in a given partnership is not affected if this agent or his partner breaks some of his agreements in other partnerships or add new agreements to the pool, or leave some of his partners.

the set of competitive equilibrium allocations may be distinct sets.[4] New characterizations for these sets and new correlations among them were established, leading to the discovery that the solution concept that captures the idea of cooperative equilibrium in matching games, called stability since Gale and Shapley (1962), can also be extended to a certain class of non-matching games. In these games, the utilities are additively separable and, instead of pairs, the agents form coalitions of any size (Sotomayor 2010).

In the present manuscript, motivated by the results obtained for the multiple-partners assignment game and for the model proposed in Sotomayor (2010), we search new characterizations for the three solutions concepts mentioned above, as well as new correlations among them, aiming to reach a better understanding of these concepts. To reach these objectives, we formulate a new model called here time-sharing assignment game. To guarantee the distinction between the core and the set of cooperative equilibrium allocations, the common assumption of the multiple-partners assignment game and of the coalitional game presented in Sotomayor (2010) is maintained: the utilities are additively separable and the individual payoffs are multi-dimensional in both sides of the market, so the trades done by any agent are independent and pairwise. Nevertheless, instead of quotas in terms of the number of partneships an agent can form, it is considered that each participant owns an amount of units of a divisible good (e.g., labor time)[5], which should be distributed among his [6] partners in any way he agrees and exchanged for money.

The negotiations inside a partnership involve two kinds of agreements: the division of the gains of the partnership among the partners and the amount of labor time each partner should contribute. The more or less flexibility between these two types of trades opens up to two closely related cooperative markets, the rigid and the flexible markets[7]. In the rigid market all agreements are rigid. This means that an agreement is

[4] The fact that the set of cooperative equilibrium allocations may be a proper subset of the core was first proved in Sotomaor (1992); that the set of competitive equilibrium allocations may be a proper subset of the set of cooperative equilibrium allocations was first proved in Sotomayor (2007).
[5] This assumption is not necessary. All of our results and concepts apply when the goods are indivisible and the amounts are integers. In this case the rules of the game of the multiple partners assignment game are slightly modified: a buyer can acquire more than one unit of the good from the same seller.

[6] For the sake of exposition we will treat any generic agent as “he”, any buyer as “she” and any seller as “he”.

[7] In the model presented in Sotomayor (2010) the agreements are always rigid due to the fact that the trades are not, necessarily, pairwise.

nullified once any of its terms is changed. In the flexible market all agreements are flexible. A flexible agreement between two partners allows either agent to decrease the number of units of labor time (u.l.t. for short) he contributes to the partnership without breaking the agreement corresponding to the division of the income per u.l.t.. Therefore, either agent is allowed to transfer part of his common labor time to some other current partnerships or to some new partnerships.

The feasible allocations specify the individual payoffs for each agent and the amount of labor time allocated to each partnership that it is formed. Intuitively, a feasible allocation is a *cooperative equilibrium* (or a *stable* allocation) if there is no coalition whose members can profitably "deviate" from the given allocation by acting according to the rules of the game. This idea contrasts with that of core in that the members of a blocking coalition "deviate" by interacting only among them. Weather the agents can do more than that, it is established by the rules of the game.

Therefore, in order to characterize the cooperative equilibrium allocations we need to make precise the rules of the game. The difficulty we face is that, unlike most of the cooperative games, the feasible allocations do not fully model the rules of the game, since they do not inform if the agreements inside each partnership are rigid or flexible. Therefore, the two markets provide the same sets of feasibe allocations. Also, the characteristic function does not capture the nature of the agreements, so the two markets are indistinguishable under their representation in the characteristic function form. As it is illustrated in the text by an example, it might happen that, given a feasible allocation, it would be of interest of the members of a coalition to transfer part of their labor time from some current partnerships to some other current partnerships or to some new partnerships. This kind of deviation is allowed in the flexible market but it is not so in the rigid market. Then, the given allocation might be a cooperative equilibrium allocation for the rigid market and it might not be so for the flexible market. In this case, the use of the characteristic function in modeling the rules of the rigid and the flexible markets would be inappropriate for the purpose of observing cooperative equilibrium allocations.

Thus, it is crucial to provide a model for the rules of these markets that captures the information on the nature of the agreements concerning the labor time. To this end, an axiomatic theory is proposed, permiting to find out a general form to fully represent the cooperative games defined by these markets. This representation is called *deviation* *function form* [8]. Once this foundation is laid out, then it becomes possible to identify the cooperative equilibrium allocations in each market.

As in every continuous two-sided matching model, another way of looking upon these markets is to think of them operating as in an exchange economy. The agents are buyers and sellers. A competitive market is specified by the set of agents and the demand correspondences for the buyers. The natural solution concept is called *competitive equilibrium allocation.*[9] By assuming that the prices of all units of labor time owned by a seller are equal,[10] a feasible allocation is a competitive equilibrium allocation if the demand of the buyers is satisfied at the given prices and all units of labor time with a positive price are sold.

We provide two different definitions for the demand correspondences of the buyers, which generate two distinct competitive markets: competitive market with discriminatory demands and competitive market with non-discriminatory demands. In both markets, given the prices, a buyer demands a feasible assignment for herself (a bundle of units of labor time that satisfies her quota). In the former market, she demands the feasible assignments which give her the maximum total surplus, assuming this number is non-negative; in the other market, she demands the feasible assignments which give her the maximum total surplus among those which complete her quota and give her the same individual surpluses, if such assignments exist, assuming these individual surpluses are non-negative. (This definition is slightly different in the text in order to include the case in which the individual gains are zero. There, dummy objects are included to fill the quotas of the buyers). Clearly, the set of competitive equilibrium allocations of the former market contains the set of competitive equilibrium allocations of the other market.

In the present work we characterize the cooperative equilibrium allocations and the competitive equilibrium allocations in the rigid and flexible markets, show their existence and establish the connection between the cooperative and competitive structures in each market.

---

[8] The deviating function form was introduced in Sotomayor (2011) to model cooperative decision situations in which agents form coalitions and freely interact inside each coalition that is formed, by acting according to some established rules.

[9] This concept was introduced in Sotomayor (2007).

[10] In the multiple partners assignment game, this is not an assumption of the model, but a consequence of the requirement that each buyer cannot acquire more than one object from the same seller.

We capitalize on the fact that the two models are treated altogether to make comparisons between their cooperative structures. Such treatment allows the identification of five cooperative solution concepts, defined in the text via three types of domination relation. These solution concepts apply to each cooperative market and have each a special meaning for the cooperative and competitive markets. Namely they are the corewise-stability, the setwise-stability, the strong-stability, the buyer-non-discriminatory strong-stability and the non-discriminatory strong-stability. Setwise-stablity is the translation of the concept defined in Sotomayor (2010). The last three concepts are newly defined. We show that the corresponding solution sets are non-empty, distinct and correlated by the set inclusion: one is a superset of the next. The maximal set is the core[11]; the set of setwise-stable allocations characterizes the set of cooperative equilibria for the rigid market; the set of strongly-stable allocations characterizes the set of cooperative equilibria for the flexible market; the set of buyer-non-discriminatory strongly stable allocations is identified with the set of cooperative equilibrium allocations of the flexible market that do not discriminate the buyers. This set characterizes the set of competitive equilibrium allocations for the competitive market under discriminatory demands. The minimal set, the set of non-discriminatory strongly-stable allocations, is identified with the set of cooperative equilibrium allocations of the flexible market that do not discriminate any agent. This set characterizes the set of competitive equilibrium allocations for the competitive market under non-discriminatory demands.

As it is the case of the multiple-partners assignment game and of the model presented in Sotomayor (2010), in the models treated here, the distinction between the core and the set of cooperative equilibria is also due to the multi-dimensionality of the agents' payoffs, in both sides of the market, and to the independence among the trades. In fact, these assumptions imply that the agents in a deviating coalition can do more than the members of a blocking coalition can do. Besides to merely trade among them, they are also allowed to renegotiate among them while keeping (or reformulating, when the agreements are flexible) the terms of some current agreements with current partners outside the group. The difference between the sets of cooperative equilibrium allocations in the rigid and in the flexible markets is due to the fact that the renegotiation of a current agreement takes into consideration the nature of the

[11] The corewise-stability captures the idea of cooperative equilibrium for the time-sharing assignment game in which all agents have one-dimensional payoff, given by their total payoff.

agreements. Only flexible agreements can be reformulated. This may generate distinct coalitional interactions in the two markets. As showed in an example given in the text, the nature of the agreements can lead to deviations that are feasible in the flexible market and are not feasible in the rigid market.

The characterizations of the sets of competitive equilibrium allocations as subsets of strongly-stable allocations establish the link between the cooperative and the competitive structures of the rigid and the flexible markets. They do not take into account the nature of the agreements. More specifically, the nature of the agreements, which generates distinct sets of cooperative equilibria, has no effect on the competitive structures, however. The competitive structure under the rigid market is the same as the one under the flexible market.

Thus, the cooperative structure of the flexible market creates a bridge between the competitive and the cooperative structures of the rigid market: the competitive equilibrium allocations are also cooperative equilibrium allocations under rigid agreements and so they are in the core. However, as it is proved here, the kind of correlation between the competitive and the cooperative equilibria is not the same in the two markets. An example illustrates that, unlike the flexible market, the cooperative equilibrium allocations which do not discriminate the buyers in the rigid market, as well as those which do not discriminate any agent, are not necessarily competitive in any of the two competitive markets. An implication of this result is that the fraction of the cooperative equilibrium allocations that are competitive is smaller under rigid agreements than under flexible agreements.

It is worth to point out that, as far as we know, the link between the cooperative and the competitive structures of the markets presented here is new. In fact, the multiple-partners assignment game and the flexible time-sharing assignment game share the property that the competitive equilibrium allocations (under discriminatory demands) are the cooperative equilibrium allocations that do not discriminate the buyers. However, the competitive equilibrium allocations for the multiple-partners assignment game can be created by "shrinking" the set of cooperative equilibrium allocations through an isotone function. The set of competitive equilibrium payoffs is then characterized as being the set of fixed points of that function. Such characterization fails to hold in the flexible market, as we see in the text through an example.

The proof of the existence theorem is obtained by showing that the competitive equilibrium allocations for the competitive market with non-discriminatory demands are exactly the dual allocations. These are the allocations that are naturally defined from the dual solutions of the transportation problem whose objective function is the value of the grand coalition. By the Duality Theorem, these dual solutions always exist, so all solution sets for the time-sharing assignment game are non-empty.

This work is organized as follows. In section 2 we present the cooperative structure of the rigid and flexible markets. In sub-section 2.1 we describe the two cooperative models and give the common definitions. Sub-section 2.2 defines the cooperative solution concepts, illustrates with some examples that the set inclusion relation among the corresponding solution sets may be strict and presents some results related to the core. In sub-section 2.3 we define the primitives of the game in the deviation function form and characterize the cooperative equilibrium allocations in both cooperative markets. Section 3 presents the competitive framework. Sub-section 3.1 describes the two competitive markets and sub-section 3.2 gives the definition of the competitive equilibrium concept. Section 4 identifies the two sets of competitive equilibrium allocations and then discusses the correlation between the cooperative and the competitive structures. Section 5 proves the existence theorem for the solution sets. Final remarks and related work are presented in section 6. The longer proofs are given in the Appendix.

2. COOPERATIVE STRUCTURE OF THE TIME-SHARING ASSIGNMENT GAME

2.1 FORMAL COOPERATIVE MODEL

There are two finite and disjoint sets of agents, P with m elements and Q with n elements, which we may think of as a set of buyers and a set of sellers, or a set of firms and a set of workers. We will describe the model in terms of buyers and sellers, who we will sometime call P-agents and Q-agents, respectively. Generic agents of P and Q will be denoted by p and q, respectively.

Each agent has a quota of units of labor time (e.g. man-hours) at his disposal, which he can distribute among the partnerships he forms in any way he likes. Each seller q supplies a quota of $s(q) \in R^+$ units of labor time (u.l.t. for short) and each buyer p cannot acquire more than her quota of $r(p) \in R^+$ u.l.t.. Dummy-agents, denoted by 0, will be included in both sides of the market for technical convenience. The quota $r(0)$ of the dummy P-agent is equal to $\sum_{q \in Q} s(q)$ and the quota $s(0)$ of the dummy Q-

agent is equal to $\sum_{p\in P} r(p)$. We will assume that the reservation price of one u.l.t. is zero for all sellers. For each pair $(p,q)\in P\text{x}Q$ there is a nonnegative number a_{pq}, which is to be split between the partners in any way they agree. The number a_{pq} can be interpreted as the maximum amount of money buyer p would consider paying for one unit of labor time supplied by seller q. Then, a_{pq} is the gain from trade when one u.l.t. of seller q is sold to buyer p. Thus, if seller q sells one u.l.t. to buyer p at price w_{pq} then p will get the individual payoff of $u_{pq}=a_{pq}-w_{pq}$ and q will receive w_{pq}. The matrix of numbers a_{pq}'s will be denoted by a, with $a_{p0}=a_{0q}=0$ for all $(p,q)\in P\text{x}Q$. The agents who are not dummies will some times be called *real agents.* We will denote by r and s, respectively, the sets of numbers $r(p)$′s and $s(q)$′s.

The rules of the game are that any $p\in P$ and $q\in Q$ may form a partnership if they both agree. If a partnership (p,q) is formed, both agents should agree about the labor time they must contribute to the partnership and the division of the income a_{pq} among them.

We will assume that for a partnership (p,q) to be active both members must contribute the same positive amount of units of labor time and each agent should receive equal individual payoff per each u.l.t. he contributes to the partnership. This assumption is natural under our buyer-seller market interpretation: if a trade between a buyer and a seller is performed then the number of units of labor time acquired by the buyer is equal to the number of units of labor time sold by the seller. Furthermore, all these u.l.t. are sold to the buyer for the same price and so the buyer gets the same individual payoffs with all of them.

The rules of the game must also specify the kind of agreement concerning the amount of labor which is to be contributed to the partnership by its members. We consider two types of agreements:

Rigid agreement - if p or q breaks the agreement regarding the amount of labor, then the whole agreement, including the division of the income, must be nullified.

Flexible agreement – a flexible agreement between p and q allows either agent to decrease the number of u.l.t. he contributes to the partnership without breaking the agreement corresponding to the division of the income per u.l.t.. Therefore, any of the two agents is allowed to transfer part of his common labor time to some other current partnerships or to some new partnerships.

Let us call *rigid market* (respectively, *flexible market)* the time sharing assignment game in which the rules require that all agreements be rigid (respectively, flexible).

The players seek to form sets of partnerships to distribute all their labor time. The obvious condition for feasibility is that all money generated by a partnership per u.l.t. is distributed among its members. Formally,

Definition 2.1.1 *A **labor time allocation** is a real matrix $x=(x_{pq})$. The labor time allocation x is **feasible** if*

(a) $\sum_{q\in Q} x_{pq}=r(p)$ *for all real agents* $p\in P$;

(b) $\sum_{p\in P} x_{pq}=s(q)$ *for all real agents* $q\in Q$.

(c) $x_{pq}\geq 0$ *for all pairs* $(p,q)\in P\text{x}Q$.

The number x_{pq} (non-necessarily integer) may be interpreted as the amount of labor time p and q work together. Note that *(a)* is an equation, not an inequality, because p can always contribute left over labor time to the partnership *(p,0)*. Similar observation applies to *(b)*.

A feasible labor time allocation x is *optimal* if

(d) $\sum_{(p,q)\in P\text{x}Q} a_{pq}\, x_{pq}\geq\sum_{(p,q)\in P\text{x}Q} a_{pq}\, x'_{pq}$, for all feasible labor time allocations x'.

NOTATION: For the labor time allocation x, set $\boldsymbol{C(x)\equiv\{(p,q)\in P\text{x}Q;\ x_{pq}>0\}}$. If $(p,q)\in C(x)$ we say that *(p,q)* is *active at* x (or simply *active*, for short, when there is no confusion). For the labor time allocation x and $(p,q)\in PxQ$, set $\boldsymbol{B(p,x)\equiv\{q'\in Q;\ (p,q')\in C(x)\}}$ and $\boldsymbol{B(q,x)\equiv\{p'\in P;\ (p',q)\in C(x)\}}$.

Definition 2.1.2. *Given a labor time allocation x, a **money allocation** (u,w) corresponding to x is a pair of non-negative real functions on $C(x)$. It is **feasible** if x is feasible and*

(e) $u_{pq}+w_{pq}= a_{pq}$ for all $(p,q)\in C(x)$.

*We say that (u,w) is compatible with x and vice-versa. The triple (u,w;x) is called a **feasible allocation.***

That is, *(u,w;x)* is a feasible allocation if it satisfies *(a), (b), (c)* and *(e)*. Condition *(e)* clearly implies that $u_{p0}=w_{0q}=0$ if the corresponding partnerships are active. Observe that $\boldsymbol{u_{pq}}$ **is not defined if** $\boldsymbol{x_{pq}=0}$.

NOTATION: **(a)**We will denote by Σ the set of all feasible allocations. A player compares two feasible allocations by comparing his total payoff in each allocation. The p's total payoff and the q's total payoff generated by $(u,w;x)$ are given, respectively, by: $U_p=\sum_{q\in B(p,x)} u_{pq}x_{pq}$ and $W_q=\sum_{p\in B(q,x)} w_{pq}x_{pq}$. **(b)** For every $p\in P$ and $q\in Q$ define $u_p(min)=min\{u_{pq};\ q\in B(p,x)\}$ *and* $w_q(min)= min\{w_{pq};\ p\in B(q,x)\}$.

Definition 2.1.3. *The feasible allocation $(u,w;x)$ is P-**non-discriminatory** if*

(f) $w_{pq}=w_q(min)$ ***for all*** $(p,q)\in C(x)$.

*The feasible allocation $(u,w;x)$ is Q-**non-discriminatory** if*

(g) $u_{pq}=u_p(min)$ ***for all*** $(p,q)\in C(x)$.

*The feasible allocation $(u,w;x)$ is **non-discriminatory** if the payoff functions u and w satisfy (f) and (g).*

Definition 2.1.4. *Let $S\subseteq P\cup Q$, $S\neq\phi$. The feasible labor time allocation x is **feasible for S** if, for every real agent $p\in S$ and real agent $q\in S$, $[B(p,x)-\{0\}]\subseteq S$ and $[B(q,x)-\{0\}]\subseteq S$. The feasible allocation $(u,w;x)$ is **feasible for S** if x is feasible for S.*

That is, under the assumptions above, x is feasible for S if no real agent in this set interacts, at x, with real agents out of S.

For every coalition S, define $V(S)$ as being the set of feasible allocations that are feasible for S. That is,

(h) $V(S)=\{(u,w;x)\in\Sigma;\ x$ **is feasible for** $S\}$.

For each $R\subseteq P$ and $T\subseteq Q$ the gain $G(R\cup T)$ of coalition $R\cup T$ is given by

(i) $G(R\cup T)\equiv max\ \{\sum_{(p,q)\in RxT} a_{pq}\, x_{pq};\ x$ *is feasible for* $R\cup T\}$.

That is, $G(R\cup T)$ is the maximum income the real players in $R\cup T$ can get by themselves. According to this definition, a labor time allocation x is optimal if and only if $G(P\cup Q)=\sum_{(p,q)\in PxQ} a_{pq}\, x_{pq}$.

REMARK 2.1.1. From Definition 2.1.4, if $(u,w;x)\in V(R\cup T)$ then the real players of $R\cup T$ achieve their total payoff and fill their quota of labor time without any interaction with real players out of $R\cup T$. The feasibility of $(u,w;x)$ then implies that $\sum_{p\in R,q\in T} U_p+W_q = \sum_{(p,q)\in RxT} a_{pq}\, x_{pq}$. By (i), $\sum_{p\in R,q\in T} (U_p+W_q)\leq G(R\cup T)$. In particular, since any feasible allocation $(u,w;x)$ belongs to $V(P\cup Q)$, $\sum_{p\in P,q\in Q}(U_p+W_q)$

$\leq$ ***G(P∪Q).*** However, it is very easy to find an allocation that satisfies this expression and does not satisfy *(e),* so it is not feasible. ■

2.2 COOPERATIVE SOLUTION CONCEPTS

In the previous sub-section we have presented the basic ingredients that describe the rigid and the flexible markets. In these markets, the payoff of an agent may be multi-dimensional and the trades are done independently and pairwisely. A third model for the time-sharing assignment game can be obtained by assuming that **agents negotiate in block** with their whole set of partners and disregard the individual payoffs they could obtain in each individual transaction. Under these rules, an outcome *(U,W;x)* would be a vector of total payoffs, one total payoff for each player, plus a labor time allocation. Within this context, the outcome *(U,W;x)* is feasible if $\sum_{p\in P,\, q\in Q}(U_p + W_q) \leq G(P\cup Q)$. This model is studied in Sotomayor (2002). We will refer to it as the **time-sharing assignment game with one-dimensional payoffs.**

In this sub-section we define the following cooperative solution concepts that apply to all three models: *corewise-stability, setwise-stability, strong-stability, P-non-discriminatory strong-stability*[12] *and non-discriminatory strong-stability.* We show that the corresponding solution sets are set inclusion related: one is a super-set of the next and the set inclusion relation may be strict. In the next sub-section we will see that corewise-stability is the cooperative equilibrium concept for the time-sharing assignment game with one-dimensional payoffs. Setwise-stability is the cooperative equilibrium concept for the rigid market and strong-stability is the cooperative equilibrium concept for the flexible market. Section 4 will show that the other two cooperative solution concepts have a relevant importance in the establishment of the link between the cooperative and the competitive structures of the rigid and flexible markets. They are exactly the concepts of competitive equilibrium allocation for two market games defined by two different demand correspondences.

Three types of domination relations are used to define the cooperative solution concepts.

[12] The *Q-non-discriminatory strong-stability* concept is defined symmetrically. Since we are treating the *P*-agents as buyers, the *P-non-discriminatory strong-stability* concept has a special interpretation in the competitive market.

Definition 2.2.1. *The feasible allocation* $\sigma'=(u',w';x')$ ***dominates*** *the feasible allocation* $\sigma=(u,w;x)$ *via coalition* $S=R\cup T\neq\phi$, *with* $R\subseteq P$ *and* $T\subseteq Q$, *if*

(i_1) $U'_p > U_p \ \ \forall p\in R$ *and* $W'_q > W_q \ \ \forall q\in T$ *and*

(i_2) $(u',w';x')\in V(S)$.

That is, the feasible allocation σ' dominates the feasible allocation σ via coalition S if every player in S prefers σ' to σ and the players of coalition S reach allocation σ' by

1. breaking all their current agreements, and
2. replacing their current agreements with a new set of agreements, which only involves players in S.

This is to say that the feasible allocation σ is dominated by another feasible allocation σ' via coalition S if the players in S can **profitably deviate** from the given allocation σ and obtain σ' **by interacting only among them**. This is how a corewise-stable allocation is defined. That is,

Definition 2.2.2. *The feasible allocation is* ***corewise-stable*** *if it is not dominated by any other feasible allocation via some coalition. Such a coalition is called* ***blocking coalition.***

The following two propositions will be useful. Proposition 2.2.1 gives a sufficient condition for a feasible allocation to be corewise-stable. Proposition 2.2.2 asserts that every corewise-stable allocation is individually rational.

Proposition 2.2.1: *Let* $(u,w;x)$ *be a feasible allocation such that*

(j) $\sum_{p\in R} U_p + \sum_{q\in S} W_q \geq G(R\cup S)$, *for every* $R\subseteq P$ *and* $S\subseteq Q$.

Then, $(u,w;x)$ *is corewise-stable.*

Proof: If the feasible allocation $(u,w;x)$ was dominated by some feasible allocation $(u',w';x')$ via some coalition $R\cup T$, Definition 2.2.1-(i_2) would imply that $(u',w';x')\in V(R\cup T)$. By Remark 2.1.1, $\sum_{p\in R} U'_p + \sum_{q\in T} W'_q \leq G(R\cup T)$. Definition 2.2.1-($i_1$) then would imply $\sum_{p\in R} U_p + \sum_{q\in T} W_q < G(R\cup T)$, which is a contradiction■.

Proposition 2.2.2. *If* $(u,w;x)$ *is corewise-stable then*

(k) $U_p \geq 0$ for all $p \in P$ *and* $W_q \geq 0$ for all $q \in Q$.

Proof. This is immediate from the fact that if, say $U_p<0$ for some agent $p \in P$, then any feasible allocation that gives zero amount of money to p and allocates her quota of labor time to the dummy *Q*-agent, would dominate *(u,w;x)* via coalition $S=\{p,0\}$. ■

The concept of setwise-stability can be defined by the quasi-domination relation. Roughly speaking, the feasible allocation σ' quasi-dominates the feasible allocation σ via coalition S if every player in S prefers σ' to σ and the players of coalition S reach allocation σ' by

1. breaking some of their agreements (non-necessarily all of them),
2. keeping the whole terms of the current agreements which were not broken, and
3. replacing the dissolved agreements with a new set of agreements which only involves players in S.[13]

Formally,

Definition 2.2.3. *Let $\sigma=(u,w;x)$ and $\sigma'=(u',w';x')$ be feasible allocations. Allocation σ'* ***quasi-dominates*** *allocation σ via coalition $R \cup T$, with $R \subseteq P$ and $T \subseteq Q$ if*

(i₁) $U'_p > U_p$ $\forall p \in R$ *and* $W'_q > W_q$ $\forall q \in T$, *and*

(i₂) *if* $[p \in R$ *and* $x'_{pq}>0]$ *then* $[q \in T]$ *or* $[x'_{pq}=x_{pq}$ *and* $u'_{pq}=u_{pq}]$; *if* $[q \in T$ *and* $x'_{pq}>0]$ *then* $[p \in R]$ *or* $[x'_{pq}=x_{pq}$ *and* $w'_{pq}=w_{pq}]$.

Definition 2.2.4. *Allocation $\sigma=(u,w;x)$ is* ***setwise-stable*** *if it is feasible and is not quasi-dominated by any feasible allocation via some coalition.*[14]

The strong quasi-domination relation can be used to define the strong stability concept. Roughly speaking, the feasible allocation σ' strongly quasi-dominates the feasible allocation σ via coalition S if every player in S prefers σ' to σ and the players of coalition S reach allocation σ' by

---

[13] The quasi-domination relation defined here is a translation to this model of the concept introduced in Sotomayor (2010) for a coalitional game, which is not a matching game.

[14] The translation of the concept of setwise-stability for the discrete matching models was first proposed in Sotomayor (1999-b). For the Multiple-partners assignment game, the setwise-stability coincides with the concept of stability.

1. breaking some of their agreements (non-necessarily all of them),

2. keeping the whole terms of some of their current agreements which were not broken and

2'. reformulating the terms of the remaining current agreements (which were not dissolved and were not maintained) with respect to the time allocation (by reducing the number of u.l.t and keeping the terms on the division of the income a_{pq}), and

3. replacing the dissolved agreements with new agreements which only involve players in S.

Formally,

Definition 2.2.5. *Let $\sigma=(u,w;x)$ and $\sigma'=(u',w';x')$ be feasible allocations. The feasible allocation $\sigma'=(u',w';x')$* ***strongly quasi-dominates*** *the feasible allocation $\sigma=(u,w;x)$ via coalition $S=R\cup T\neq\phi$, with $R\subseteq P$ and $T\subseteq Q$, if*

(i_1) $U'_p>U_p$ $\forall p\in R$ and $W'_q>W_q$ $\forall q\in T$, and

(i_2) if $[p\in R$ and $x'_{pq}>0]$ then $[q\in T]$ or $[x_{pq}\geq x'_{pq}$ and $u'_{pq}=u_{pq}]$; if $[q\in T$ and $x'_{pq}>0]$ then $[p\in R]$ or $[x_{pq}\geq x'_{pq}$ and $w'_{pq}=w_{pq}]$.

Definition 2.2.6. *A feasible allocation is* ***strongly-stable*** *if it is not strongly quasi-dominated by any other feasible allocation via some coalition.*

Therefore, domination implies quasi-domination, which implies strong quasi-domination. Thus, the core contains the set of setwise-stable allocations, which contains the set of strongly stable allocations. Indeed, as the examples below show, all these inclusions may be strict.

Example 2.2.1. (The core is bigger than the set of setwise-stable allocations) Consider $P=\{p_1,p_2\}$, $Q=\{q_1, q_2\}$ $r(p_1)=r(p_2)=s(q_2)=2$ and $s(q_1)=1$. The numbers a_{pq}'s are given by: $a_{11}=3$, $a_{21}=5$, $a_{12}=2$, $a_{22}=3$. The nature of the agreements is arbitrary. Consider the allocation $(u,w;x)$ where $x_{11}=0$, $x_{12}=1$, $x_{10}=1$, $x_{21}=x_{22}=1$, $x_{20}=0$ and $u_{12}=1$, $u_{10}=0$, $u_{22}=1$, $u_{21}=3$; $w_{12}=1$, $w_{21}=2$, $w_{22}=2$. The corresponding total payoffs are $U_1=1$, $U_2=4$, $W_1=2$ and $W_2=3$.

The values of the coalitions are given by: $G(p_1,q_1)=3$, $G(p_1,q_2)=4$, $G(p_2,q_1)=5$, $G(p_2,q_2)=6$, $G(p_1,q_1,q_2)=5$, $G(p_2,q_1,q_2)=8$, $G(p_1,p_2,q_1)=5$, $G(p_1,p_2,q_2)=6$,

$G(p_1,p_2,q_1,q_2)=10$, $G(S)=0$ if $S \subseteq P$, or $S \subseteq Q$. It is a matter of verification that *(j)* is satisfied. Proposition 2.2.1 then implies that $(u,w;x)$ is corewise-stable. However, $(u,w;x)$ is not setwise-stable. In fact, players p_1 and q_1 can increase their total payoffs if p_1 keeps his agreement with q_2, q_1 breaks his agreement with p_2, p_1 and q_1 work together 1 u.l.t. and receive for this labor, respectively, 0.5 and 2.5. ■

Example 2.2.2. (The set of setwise-stable allocations is bigger than the set of strongly-stable allocations) Consider $P=\{p_1\}$, $Q=\{q_1, q_2\}$, $r(p_1)=5=s(q_1)$, $s(q_2)=1$, $a_{11}=a_{12}=3$. The nature of the agreements is arbitrary. Consider the allocation $(u,w;x)$ where $x_{11}=5, x_{12}=0, x_{02}=1; u_{11}=1, w_{11}=2, w_{02}=0$. Then $U_1=5$, $W_1=10$ and $W_2=0$.

Allocation $\sigma=(u,w;x)$ is setwise-stable, so it is corewise-stable. It is easy to verify that there is no way for p_1 to increase his total payoff by only trading with q_2. In order to increase his total payoff, p_1 must trade with both sellers, but there are no prices that can increase the current total payoffs of the three agents.

Allocation σ **is not strongly-stable**. In fact, the feasible allocation $\sigma'=(u',w';x')$, where $x'_{11}=4<5=x_{11}$, $x'_{12}=1$, $u'_{11}=1$, $u'_{12}=2$, $w'_{11}=2$, $w'_{12}=1$, is preferred to σ by p_1 and q_2. The agreement between p_1 and q_2 at σ' is new and the trade between p_1 and q_1 is a reformulation of the current trade: The number of negotiated units of labor time is reduced to 4, but the division of the income a_{11} between the agents is maintained. ■

These two examples also illustrate that the interactions allowed among the members of a coalition, for the purpose of blocking an allocation, are not affected by the nature of the agreements.

The *P*-non-discriminatory strong-stability and non-discriminatory strong-stability concepts are naturally defined.

Definition 2.2.7 *A feasible allocation is* ***P-non-discriminatory strongly stable*** *if it is strongly stable and P-non-discriminatory; a feasible allocation is* ***non-discriminatory strongly stable*** *if it is strongly stable and non-discriminatory.*

Of course, the set of strongly stable allocations contains the set of *P*-non-discriminatory strongly stable allocations, which contains the set of non-discriminatory strongly stable allocations. It is easy to construct examples in which these inclusions are strict.

2.3. COOPERATIVE EQUILIBRIUM AND THE DEVIATION FUNCTION FORM

Since Gale and Shapley (1962), the cooperative equilibrium allocations in matching markets are called stable allocations. The general idea proposed in Sotomayor (2011) is that *an allocation is* ***stable*** *if there is no coalition of players who can profitably deviate from the given allocation* ***by acting according to the rules of the game.*** *Such a coalition is called deviating coalition.*

This concept differs from that of corewise-stability in markets where the players of a coalition are allowed to do more than to merely interact among themselves. What agents can do must be specified by the rules of the game.

In the time-sharing assignment games with multi-dimensional payoffs, the independence among the trades, implied by the assumption that the utilities are additively separable, means that the rules of these markets allow that agents in a coalition can renegotiate among themselves while keeping the terms of some current agreements with current partners outside the group. However, the trades inside a partnership take into consideration the nature of the agreements, so the rules of the rigid and flexible markets are distinct. It is a fact that the nature of the agreements may generate coalitional interactions that are relevant for the purpose of observing cooperative equilibrium allocations. Indeed, such coalitional interactions can lead to new types of deviations which might destabilize some corewise-stable allocations. In Example 2.2.2, for instance, the agreement between p and q_2 at σ' is new, but the agreement between p and q_1 at σ' is a reformulation of the agreement between these agents at σ. Therefore, if we do not know if this kind of interaction is allowed by the rules of the game, we cannot predict if σ will or will not occur. Allocation σ is stable in the rigid market but it is unstable in the flexible market.

How to model the rules of the game so that to capture the information on the nature of the agreements?

It is worth to point out that, unlike the continuous matching models studied in the literature, the coalitional function $V,$ defined here for each coalition S as the set of feasible allocations which are feasible for $S,$ is not suitable to fully represent the time-sharing assignment game with multi-dimensional payoffs, whatever kind of agreement is being considered. In the situation illustrated in Example 2.2.2, for instance, there is no way to capture the nature of the agreements from the characteristic function V. This function only informs that σ is in $V(P\cup Q)$ and it is not in $V(S)$. Thus, in general, we cannot guarantee that an allocation is stable by only using the characteristic function V.

Therefore, it becomes more convenient to work with the *deviation function form* of the game. This representation was introduced in Sotomayor (2011) and it is given by the set of agents, the set of feasible allocations and for each feasible allocation σ and coalition S, the set of *feasible deviations from* σ *via coalition* S.

Roughly speaking, a feasible allocation σ' is a *feasible deviation from* σ *via coalition* S *for the rigid market* if the players in S obtain σ' from σ by breaking some or all of their agreements at σ, by keeping those ones at σ which were not broken and by replacing the broken agreements at σ with a new set of agreements, which only involves agents in S. Therefore, the players in S obtain σ' by modifying σ through actions allowed by the rules of the game that take into account the rigid nature of the agreements. Formally,

Definition 2.3.1. *Given a coalition* $S\subseteq P\cup Q$ *and a feasible allocation* $\sigma=(u,w;x)$, *a feasible allocation* $\sigma'=(u',w';x')$ *is a* ***feasible deviation from*** σ ***via*** S *for the rigid market if*

(l) *when* [$p\in S$, $x'_{pq}>0$ *and* ($x'_{pq}\neq x_{pq}$ *or* $u'_{pq}\neq u_{pq}$)] *then* $q\in S$; *when* [$q\in S$, $x'_{pq}>0$ *and* ($x'_{pq}\neq x_{pq}$ *or* $w'_{pq}\neq w_{pq}$)] *then* $p\in S$;

(m) *for every* p *in* S, *there is some* q *in* S *such that* $x'_{pq}>0$ *and* [$x'_{pq}\neq x_{pq}$ *or* $u'_{pq}\neq u_{pq}$]; *and for every* q *in* S, *there is some* p *in* S *such that* $x'_{pq}>0$ *and* [$x'_{pq}\neq x_{pq}$ *or* $w'_{pq}\neq w_{pq}$].

If σ' *is a feasible deviation from* σ *via* S *we say that* S *is a* ***deviating coalition***.

When agreements are flexible, a deviating coalition can do more than the rules specify when agreements are rigid. In fact, a feasible allocation σ' is a *feasible deviation from* σ *via coalition* S *for the flexible market* if the players in S obtain σ' from σ by breaking some or all of their agreements, keeping the whole terms of some of their current agreements which were not broken and reformulating the terms of the remaining current agreements (which were not dissolved and were not maintained) with respect to the time allocation (by reducing the number of u.l.t and keeping the terms on the division of the income a_{pq}), and replacing the dissolved agreements with new agreements that only involve agents in S. Formally,

Definition 2.3.2. *Given a coalition $S \subseteq P \cup Q$ and a feasible allocation $\sigma=(u,w;x)$, a feasible allocation $\sigma'=(u',w';x')$ is a* ***feasible deviation from σ via*** *S for the flexible market if*

(n) *when $[p \in S,\ x'_{pq}>0$ and $q \notin S]$ then $[x_{pq} \geq x'_{pq}$ and $u'_{pq}=u_{pq}]$; when $[q \in S,\ x'_{pq}>0$ and $p \notin S]$ then $[x_{pq} \geq x'_{pq}$ and $w'_{pq}=w_{pq}]$;*

(o) *for every p in S, there is some q in S such that $x'_{pq}>0$ and $x'_{pq}>x_{pq}$ or $u'_{pq} \neq u_{pq}$; and for every q in S, there is some p in S such that $x'_{pq}>0$ and $x'_{pq}>x_{pq}$ or $w'_{pq} \neq w_{pq}$.*

We can define,

Definition 2.3.3. *The allocation $\sigma \in \Sigma$ is* ***stable*** *for market M if in M there is no feasible deviation σ' from σ via some coalition S such that every player in S prefers σ' to σ.*

By Definitions 2.3.1 and 2.3.2, any feasible deviation from some $\sigma \in \Sigma$ via some coalition S for the rigid market is also a feasible deviation from σ via S for the flexible market. Therefore, the stable allocations for the flexible market are stable for the rigid market.

Let ϕ^R (respectively, ϕ^F) be the set of all feasible deviations from feasible allocations via some coalition for the rigid market (respectively, flexible market). The deviation function form of the time-sharing assignment game with rigid agreements is then given by (P,Q,Σ,ϕ^R) and for the time-sharing assignment game with flexible agreements is then given by (P,Q,Σ,ϕ^F).

The following proposition is straightforward.

Proposition 2.3.1. *a) A feasible allocation σ is stable under rigid agreements if and only if it is not quasi-dominated by any other feasible allocation via some coalition. b) A feasible allocation σ is stable under flexible agreements if and only if it is not strongly quasi-dominated by any other feasible allocation via some coalition.*

Thus, the solution concept that captures the idea of stability for the time-sharing assignment game with rigid agreements is that of setwise-stability and for the time-sharing assignment game with flexible agreements is that of strong-stability .

Of course, every blocking coalition is a deviating coalition for both, the rigid and the flexible markets although the converse is not always true. As observed before, Example 2.2.2. illustrates that an allocation may be stable under rigid agreements, unstable under flexible agreements and corewise-stable in both markets.

The corewise-stability concept is equivalent to the stability concept in the time-sharing assignment game with one-dimensional payoffs. In fact, as it is shown in Sotomayor (2002), Definition 2.2.2 is equivalent to require that $\sum_{p\in R} U_p + \sum_{q\in S} W_q \geq G(R\cup S)$, for every $R\subseteq P$ and $S\subseteq Q$, and $\sum_{p\in P,\, q\in Q}(U_p + W_q) = G(P\cup Q)$. Therefore, the characteristic function V captures all details of the rules of the game that are relevant to the model. Then, $V(S)$ equals the set of the feasible deviations from σ via S, for all $\sigma\in\Sigma$ and all $S\subseteq P\cup Q$. Hence, the corewise-stability concept is equivalent to the cooperative equilibrium concept for that model.

3. COMPETITIVE STRUCTURE OF THE TIME-SHARING ASSIGNMENT GAME

3.1 THE COMPETITIVE MARKETS

The cooperative market corresponds to situations in which an individual or group of individuals is working cooperatively toward the achievement of some well-defined goal. In the competitive market, an individual or group of individuals is not only working toward different goals but are actually competing with each other. In this section we will analyze the competitive structure of the time-sharing assignment game with multi-dimensional payoffs.

We will be assuming that all u.l.t. are supplied by the sellers. Therefore, to be well defined, the competitive market should specify the set of goods, the set of agents and the demand correspondence of each buyer. Every *seller* wants to sell his units of labor to the *buyers* and all his units of labor have the same price (the *sellers* do not discriminate the *buyers*). In this context, the prices of the goods are not negotiated, but taken as given by the buyers who, according to their demand correspondences, demand a set of bundles of units of services which respects their quotas. Our main problem is to determine how the goods will be allocated to the buyers. The natural solution concept is called *competitive equilibrium allocation*, which, informally, is a feasible allocation under which the bundle of goods allocated to a buyer belongs to her demand set at the given prices and all units of labor with a positive price are sold.

We provide two different definitions for the demand sets of the buyers, which generate two distinct competitive markets, namely, *competitive market with discriminatory demands* and *competitive market with non-discriminatory demands*.

We will illustrate these notions by using a simpler competitive market which is obtained when the goods are indivisible. In this case, every seller q supplies $s(q)$ identical objects. In the *competitive market with discriminatory demands,* denote by Q^* the set of all objects in the Economy (including the dummy objects). The prices of all objects in Q^* are announced, so that the objects supplied by a seller have the same price. It is then natural to expect that a buyer p will demand the bundles of the $r(p)$ most preferred objects in Q^* at prices p. These are the sets of $r(p)$ objects that maximize p´s total surplus among all subsets of Q^* with $r(p)$ objects, assuming this total surplus is non-negative. Of course, the objects of the demanded bundles by a buyer may produce distinct individual surpluses. The presence of the dummy objects in the Economy causes the demand sets to be non-empty.

In the *competitive market with non-discriminatory demands*, each seller q supplies $s(q)$ objects of type q (we identify seller q with the type of the objects supplied by him). Then the set of all types can be denoted by Q. The prices of all types are announced. Buyer p will demand the bundles of types of objects that are feasible for her (that respect her quota). Furthermore, in any demanded bundle, every type whose number of units in the bundle is positive, maximizes p's individual surpluses.

We extend these notions to the case where the goods are divisible. To do that, let a **feasible assignment vector for** p *(*or **assignment vector for** ***p,*** for short) be a vector of non-negative numbers $x_p \equiv (x_{pq})_{q \in Q}$ which satisfies ***(a)*** and such that $x_{pq} \leq s(q)$ for all $q \in Q$. The set of all feasible assignment vectors for p will be denoted by X_p.

Clearly, if x is a feasible labor time allocation then x_p is a feasible assignment vector for p, for all $p \in P$.

A vector $\pi \in R^n{}_+$ is called feasible price vector or price vector, for short. That is, a price vector is a vector π of non-negative numbers, one coordinate for each Q-agent, where π_q is the price of each u.l.t. offered by agent q.

3.1.1. COMPETITIVE MARKET WITH DISCRIMINATORY DEMANDS

In the competitive market with discriminatory demands, buyers have preferences over feasible assignment vectors. Given a price vector π, the preferences of agent p over feasible assignment vectors are completely described by the numbers a_{pq}*'s*. For any two assignment vectors for p, x_p and x'_p, **p prefers x_p to x'_p at prices π** if $\sum_{q\in Q} (a_{pq}-\pi_q)x_{pq} > \sum_{q\in Q} (a_{pq}-\pi_q)x'_{pq}$. Agent p is indifferent between these two assignment vectors at prices π if $\sum_{q\in S} (a_{pq}-\pi_q)x_{pq} = \sum_{q\in S'} (a_{pq}-\pi_q)x'_{pq}$. The units of labor time supplied by q are **acceptable** to p at prices π if $a_{pq}-\pi_q \geq 0$.

Under the structure of preferences we are assuming, given a price vector π, each buyer p is able to determine which assignment vectors he would most prefer. The set of such assignment vectors is called ***demand set of p at prices π*** and denoted by $D_p(\pi)$. That is,

$$D_p(\pi)=\{x_p \in X_p;\ \sum_{q\in Q}(a_{pq}-\pi_q)x_{pq} \geq \sum_{q\in Q}(a_{pq}-\pi_q)x'_{pq} \ \ \forall\ x'_p \in X_p\}.$$

Note that $D_p(\pi)$ is never empty, because p has always the option of buying the assignment vector x_p, with $x_{pq}=0$ for all $q\neq 0$ and $x_{p0}=r(p)$. Note also that, if $x_p \in D_p(\pi)$ and $x_{pq}>0$ then the units of labor time offered by q are acceptable to p.

REMARK 3.1.1. If $x_p \in D_p(\pi)$ then $a_{pq}-\pi_q \geq a_{pt}-\pi_t$ for all sellers q and t such that $x_{pq}>0$ and $x_{pt}=0$. In fact, define the feasible assignment vector x'_p, where $x'_{pq^*}=x_{pq^*}$ for all $q^* \notin \{q,t\}$, $x'_{pq}=x_{pq}-\lambda \geq 0$, $x'_{pt}=\lambda$, where $\lambda>0$. Now use the definition of $D_p(\pi)$. ■

3.1.2. COMPETITIVE MARKET WITH NON-DISCRIMINATORY DEMANDS

In the competitive market with non-discriminatory demands, buyers have preferences over the u.l.t. supplied by the sellers. Under a competitive equilibrium, the bundle of goods allocated to buyer p is a feasible assignment vector for p and it belongs to the demand set of the buyer at the given prices. Thus, for the purpose of analyzing competitive equilibria, there will be no loss in restricting the demand set of a buyer to the bundles of goods that are feasible assignment vectors for the buyer. Therefore, in the competitive market with non-discriminatory demands, given a price vector π, each buyer p will demand the assignment vector x_p if, for all $q \in Q$ with $x_{pq}>0$ and $q' \in Q$, we have that $(a_{pq}-\pi_q) \geq (a_{pq'}-\pi_{q'})$. Thus, buyer p will receive equal surpluses with all $q \in Q$ with $x_{pq}>0$.

Set $ND_p(\pi)$ the demand set of buyer p at prices π in the competitive market with non-discriminatory demands. That is,

$$ND_p(\pi)=\{x_p \in X_p;\ (a_{pq}-\pi_q) \geq (a_{pq'}-\pi_{q'})\ \ \forall\ q \in Q \text{ with } x_{pq}>0 \text{ and } q' \in Q\}.$$

Under this definition, the set $ND_p(\pi)$ may be empty.

Remark 3.1.2. Clearly, $ND_p(\pi) \subseteq D_p(\pi)$. Furthermore, if $x_p \in D_p(\pi)$ and $(a_{pq}-\pi_q) = (a_{pq'}-\pi_{q'})$, $\forall q, q' \in Q$ with $x_{pq}>0$ and $x_{pq'}>0$, then $x_p \in ND_p(\pi)$. ■

3.2. COMPETITIVE EQUILIBRIUM

The natural solution concept for the competitive markets is that of *competitive equilibrium.*

Definition 3.2.1. *The pair* (π,x) *is a* ***competitive equilibrium*** *if* (i_1) π *is a price vector,* (i_2) x *is a feasible labor time allocation such that* $x_p \in D_p(\pi)$ *for all* $p \in P$ *and* (i_3) $\pi_q=0$ *if* $x_{0q}>0$. *If* (π,x) *is a* ***competitive equilibrium*** *then* π *is called a* ***competitive equilibrium price vector (*** *or equilibrium price for short)***.**

Let x be a feasible labor time allocation satisfying condition (i_2) of Definition 3.2.1 for a price vector π. We say that π is a *competitive price vector* and x is *compatible with* π. Labor allocation x is called *competitive* if it is compatible with a competitive equilibrium price.

If (π,x) is a competitive equilibrium, the corresponding money allocation for the Q-agents, that will also be denoted by π, is defined by $\pi_{pq}=\pi_q$ for all $(p,q) \in C(x)$. The corresponding money allocation for the P-agents is defined feasibly. The resulting feasible allocation $(u,\pi;x)$ is called a **competitive equilibrium allocation** and (u, π) is called a **competitive equilibrium payoff.**

It follows from Definition 2.1.3 that a competitive equilibrium allocation for both competitive markets in consideration is P-non-discriminatory, and a competitive equilibrium allocation for the competitive market with non-discriminatory demands is a non-discriminatory allocation.

Remark 3.2.1. Since $ND_p(\pi) \subseteq D_p(\pi)$ by Remark 3.1.2, it follows that every competitive equilibrium for the competitive market with non-discriminatory demands is a competitive equilibrium for the competitive market with discriminatory demands. If a competitive equilibrium for the competitive market with discriminatory demands is a non-discriminatory allocation, then Remark 3.1.2 implies that the allocation is a competitive equilibrium for the competitive market with non-discriminatory demands. ■

4. CORRELATION BETWEEN THE COOPERATIVE AND THE COMPETITIVE STRUCTURES

The main results of this section, Theorem 4.5 and Theorem 4.6, establish the links between the cooperative and competitive structures of the rigid and flexible markets. Theorem 4.5 characterizes the competitive equilibrium allocations for the competitive market under discriminatory demands as the stable allocations of the flexible market that do not discriminate the buyers. Theorem 4.6 identifies the competitive equilibrium allocations for the competitive market under non-discriminatory demands as being the stable allocations of the flexible market which do not discriminate any agent. Therefore, the cooperative structure of the flexible market creates a bridge between the competitive and the cooperative structures of the time-sharing assignment game with rigid agreements: the competitive equilibrium allocations are also stable allocations under rigid agreements and so they are corewise-stable. However, the correlation between the cooperative and competitive structures is not the same in both markets. Example 4.1 illustrates that in the rigid market, the stable allocations which do not discriminate the buyers, as well as those which do not discriminate any agent, are not necessarily competitive in any of the two competitive markets. Thus, the fraction of the stable allocations that are competitive turns out to be smaller under rigid agreements than under flexible agreements.

Example 4.2 illustrates that the kind of correlation between the competitive equilibrium allocations and the stable allocations, in both markets, is different from that kind found in the multiple-partners assignment game.

To prove these results some preliminaries are in order.

Definition 4.1. *The pair (p,q) is* ***unsaturated with respect to the labor allocation x*** *(unsaturated, for short) if $x_{pq}<r(p)$ and $x_{pq}<s(q)$. (i.e. no player in $\{p,q\}$ contributes all his labor time to the partnership).*

In particular, if $x_{pq}=0$ then $\{p,q\}$ is unsaturated.

NOTATION: Let $(u,w;x)$ be a feasible allocation. For every unsaturated pair (p,q), define $u_{p(q)}(min)\equiv min\{u_{pr};\ r\in B(p,x)-\{q\}\}$ **and** $w_{(p)q}(min)\equiv min\{w_{tq};\ t\in B(q,x)-\{p\}\}$.

Definition 4.2. *The feasible allocation $(u,w;x)$ is* ***pairwise-strongly-stable*** *if it is feasible and*

(p) $u_{p(q)}(min) + w_{(p)q}(min) \geq a_{pq}$ *for all unsaturated pair $(p,q)\in P\text{x}Q$.*

Given a feasible allocation $(u,w;x)$ and a labor time allocation x', we can construct a feasible allocation $(u',w';x')$ so that each agent q maintains his individual payoffs in the partnerships where he decreases or keeps his labor time contribution; if q increases his labor time contribution in (p,q) then, he obtains, for each unit of additional labor time, the minimum individual payoff among all individual payoffs he obtains with partners other than p. Call F the set of such feasible allocations derived from $(u,w;x)$. Of course, $(u,w;x)$ is in F. Also, if p is distinct from p', the feasible allocation in F that maximizes p's total payoff may be different from the feasible allocation in F that maximizes the total payoff of p'. However, Theorem 4.3 asserts that this is not the case if $(u,w;x)$ is strongly-stable. Moreover, $(u,w;x)$ is strongly-stable if and only if, for all $p\in P$, $U_p=\sum_{q\in B(p,x)}u_{pq}x_{pq}$, is the highest p's total payoff that can be generated by some feasible allocation in F. By symmetry, this theorem holds if we reverse the roles between P-agents and Q-agents. To prove this result we first characterize the pairwise-strongly-stable allocations as the feasible allocations where the total payoff of every buyer is a maximum among all feasible allocations in F. Then we characterize the stable allocations of the flexible market as the pairwise-strongly-stable allocations.

Proposition 4.1. *Let $(u,w;x)$ be a feasible allocation. Then $(u,w;x)$ is pairwise-strongly-stable if and only if for all $p\in P$ and feasible labor allocation x' we have that*

(*) $U_p \geq \sum_{q\in B(p,x')}(a_{pq}-w'_{pq})x'_{pq}$,

where $w'_{pq}x'_{pq} = w_{pq}x'_{pq}$ *if* $x_{pq} \geq x'_{pq}$, $w'_{pq}x'_{pq} = w_{pq}x_{pq} + w_{(p)q}(min)(x'_{pq}-x_{pq})$ *if* $0<x_{pq}<x'_{pq}$ *and* $w'_{pq}x'_{pq}= w_{(p)q}(min)x'_{pq}$ *if* $0=x_{pq}< x'_{pq}$.

Proposition 4.2. *Let* $(u,w;x)$ *be a feasible allocation. Then* $(u,w;x)$ *is strongly-stable if and only if it is pairwise-strongly-stable.*

It follows immediately from these two propositions that:

Theorem 4.3. *Let* $(u,w;x)$ *be a feasible allocation. The following assertions are equivalent*

(i_1) $(u,w;x)$ *is strongly-stable;*

(i_2) $(u,w;x)$ *is pairwise-strongly-stable;*

(i_3) for all $p\in P$ *and feasible labor allocation* x' *we have that*

(*) $U_p \geq \sum_{q\in B(p,x')}(a_{pq}-w'_{pq})x'_{pq}$,

where $w'_{pq}x'_{pq} = w_{pq}x'_{pq}$ *if* $x_{pq} \geq x'_{pq}$, $w'_{pq}x'_{pq} = w_{pq}x_{pq} + w_{(p)q}(min)(x'_{pq}-x_{pq})$ *if* $0<x_{pq}<x'_{pq}$ *and* $w'_{pq}x'_{pq}= w_{(p)q}(min)x'_{pq}$ *if* $0=x_{pq}< x'_{pq}$.

Corollary 4.4. *Let* $(u,w;x)$ *be an allocation that is feasible and P-non-discriminatory. Then* $(u,w;x)$ *is strongly-stable if and only if, for all* $p\in P$ *and feasible labor allocation* x'*, we have*

(**) $U_p \geq \sum_{q\in B(p,x')}(a_{pq}-w_{pq})x'_{pq}$.

Proof. Immediate from Theorem 4.3.■

Corollary 4.4 implies that, under a strongly-stable allocation that is *P*-non-discriminatory, every buyer is maximizing his total payoff by taking as given the prices of the u.l.t. supplied by the sellers. This is precisely how the concept of competitive equilibrium allocation is defined when the demands are discriminatory. Then, the competitive equilibrium allocations for the market with discriminatory demands are the strongly-stable allocations such that no *Q*-agent discriminates any *P*-agent. Therefore, we have proved Theorem 4.5 below.

Theorem 4.5. *Let* $(u,w;x)$ *be a feasible allocation. Then* $(u,w;x)$ *is a competitive equilibrium allocation for the market with discriminatory demands if and only if it is strongly-stable and* $w_{pq}=w_q(min)$ for all $(p,q)\in P\text{x}Q$.

It follows from Remark 3.2.1 that if allocation σ is a competitive equilibrium allocation for the competitive market with non-discriminatory demands then it is a competitive equilibrium allocation for the competitive market with discriminatory demands. Theorem 4.5 then implies that σ is strongly-stable and no *Q*-agent discriminates any *P*-agent. Since, at σ, each buyer receives the same payoffs at all individual trades, we have that σ is a non-discriminatory strongly-stable allocation. Conversely, if σ is a non-discriminatory strongly-stable allocation, then it is *P*-non-discriminatory strongly-stable and so Theorem 4.5 implies that it is a competitive equilibrium allocation for the market with discriminatory demands. Since σ is a non-discriminatory allocation, it follows from Remark 3.2.1 that σ is also a competitive equilibrium allocation for the market with non-discriminatory demands. Then, the competitive equilibrium allocations for the market with non-discriminatory demands are the strongly-stable allocations such that no agent is discriminated. Thus we have proved the following:

Theorem 4.6. *Let* $(u,w;x)$ *be a feasible allocation. Then* $(u,w;x)$ *is a competitive equilibrium allocation for the market with non-discriminatory demands if and only if it is strongly-stable and* $u_{pq}=u_p(min)$, $w_{pq}=w_q(min)$ for all $(p,q)\in P\text{x}Q$.

The characterizations given by Theorems 4.5 and 4.6 do not take into account the nature of the agreements. More specifically, the nature of the agreements, which generates distinct cooperative structures in the time-sharing assignment game with multi-dimensional payoffs, does not have any effect on the competitive structures treated here. The following corollary follows immediately from Theorems 4.5 and 4.6 and from the fact that every strongly-stable allocation is setwise-stable.

Corollary 4.7. a) *The competitive equilibrium allocations for the market with discriminatory demands are setwise-stable (and so they are corewise-stable).*

b) *The competitive equilibrium allocations for the market with non-discriminatory demands are setwise-stable (and so they are corewise-stable).*

Example 4.1 illustrates that in the rigid market, the stable allocations which do not discriminate the buyers, as well as those which do not discriminate any agent, are not necessarily competitive in any of the two competitive markets.

Example 4.1. (Example 2.2.2 continued) (A setwise-stable allocation that is non-discriminatory – and so it is *P*-non-discriminatory – but it is not a competitive equilibrium allocation in any of the two competitive markets) Consider $P=\{p_1\}$, $Q=\{q_1, q_2\}$, $r(p_1)=5=s(q_1)$, $s(q_2)=1$, $a_{11}=a_{12}=3$. The allocation $(u,w;x)$ where $x_{11}=5$, $x_{12}=0$, $x_{02}=1$; $u_{11}=1$, $w_{11}=2$, $w_{02}=0$ is clearly non-discriminatory. As already proved, $(u,w;x)$ is setwise-stable and is not strongly-stable. Then, by Theorem 4.6, $(u,w;x)$ is not a competitive equilibrium allocation for the market with non-discriminatory demands. Since $(u,w;x)$ is also P-nondiscriminatory, Theorem 4.5 implies that $(u,w;x)$ is not a competitive equilibrium allocation for the market with discriminatory demands. ■

The kind of correlation between the competitive equilibrium allocations and the stable allocations, in both markets, is new. In the multiple partners assignment model the competitive equilibrium allocations can be created by "shrinking" the set of cooperative equilibrium allocations through an isotone function g, which maps every stable allocation $(u,w;x)$ to a competitive equilibrium allocation $(u',w';x)$ where $w'_{pq}=w_q(min)$ for all $(p,q)\in P\times Q$ and u' is feasibly defined. The set of competitive equilibrium payoffs is characterized as being the set of fixed points of that function. Such characterization fails to hold in the time-sharing assignment game with flexible agreements, as we can see in the example below.

Example 4.2. (A competitive allocation that cannot be derived from a strongly stable allocation by reducing the individual payoffs of each seller to their minimum) Consider $P=\{p_1,p_2\}$, $Q=\{q_1\}$, $r(p_1)=5=s(q_1)$, $r(p_2)=1$, $a_{11}=3$, $a_{21}=4$. The allocation $(u,w;x)$, where $x_{11}=4$, $x_{21}=1$, $x_{10}=1$; $u_{11}=1$, $u_{10}=0$, $u_{21}=1$, $w_{11}=2$, $w_{21}=3$, is clearly strongly-stable. However, the allocation $(u',w';x)$, where $w'_{11}=w'_{21}=2=min\{2,3\}$, $u'_{11}=1$, $u'_{10}=0$, $u'_{21}=2$, is not competitive in both competitive markets, since p_1 demands the whole amount of u.l.t. supplied by the seller. (Indeed this allocation is not in the core, since it is blocked by $\{p_1, q_1\}$). ■

5. NON-EMPTINESS OF THE SOLUTION SETS

Set $P^* \equiv P - \{0\}$ and $Q^* \equiv Q - \{0\}$. Consider the primal linear programming problem (P1) of finding a matrix $x=(x_{pq})$ which maximizes

(A1) $\sum_{(p,q) \in PxQ} a_{pq} x_{pq}$

subject to:

(A2) $\sum_{q \in Q^*} x_{pq} \leq r(p)$ for all $p \in P^*$;

(A3) $\sum_{p \in P^*} x_{pq} \leq s(q)$ for all $q \in Q^*$;

(A4) $x_{pq} \geq 0$ for all $(p,q) \in P^* \times Q^*$,

The dual problem (P1)* is to find an m-vector $y=(y_p)_{p \in P^*}$ and an n-vector $z=(z_q)_{q \in Q^*}$ which minimizes

(B1) $\sum_{p \in P^*} r(p) y_p + \sum_{q \in Q^*} s(q) z_q$

subject to:

(B2) $y_p + z_q \geq a_{pq}$, for all $(p,q) \in P^* \times Q^*$;

(B3) $y_p \geq 0$, $z_q \geq 0$, for all $(p,q) \in P^* \times Q^*$.

Because we know that (P1) has a solution, we know that (P*1) must have an optimal solution[15]. By the Duality Theorem, for every solution x of (P1) and (y,z) of (P1*) we have that

$$\sum_{p \in P^*} r(p) y_p + \sum_{q \in Q^*} s(q) z_q = \sum_{PxQ} a_{pq} x_{pq} = v(P \cup Q).$$

If x^* is any optimal solution for (P1) and (y,z) is an optimal dual solution then, by the Linear Programming Equilibrium Theorem or by the Complementary Slackness (see Gale, 1960), we can conclude that

(A) if $\sum_{q \in Q^*} x^*_{pq} < r(p)$ then $y_p = 0$;

(B) if $\sum_{p \in P^*} x^*_{pq} < s(q)$ then $z_q = 0$;

(C) if $x^*_{pq} = 0$ then $y_p + z_q \geq a_{pq}$;

(D) if $x^*_{pq} > 0$ then $y_p + z_q = a_{pq}$.

Now, let x^* be an optimal solution for the linear programming problem (P1) and let (y,z) be a dual optimal solution. Let x be a labor time allocation obtained from x^* as follows: $x_{pq} = x^*_{pq}$ if $p \in P^*$ and $q \in Q^*$; if $\sum_{q \in Q^*} x^*_{pq} = k < r(p)$ for some $p \in P^*$

[15] Thompson (1980) considers a model in which the core is defined as the set of dual solutions of P1.

(respectively, $\sum_{p\in P^*} x^*_{pq} = k < s(q)$ for some $q \in Q^*$) then set $x_{p0} = r(p)\text{-}k$ (respectively, $x_{0q} = s(q)\text{-}k$). Define for all $p \in P^*$ and $q \in Q^*$: $u_{pq} = y_p$ and $w_{pq} = z_q$ if $x_{pq} > 0$; $u_{0q} = w_{0q} = 0$ if $x_{0q} > 0$ and $u_{p0} = w_{q0} = 0$ if $x_{p0} > 0$. Then, by (A) and (B) $u_p(min) = y_p$ and $w_q(min) = z_q$ for all $p \in P^*$ and $q \in Q^*$. The resulting allocation $(u,w;x)$ will be called *dual allocation* and (u,w) is called dual money allocation. Then, **dual allocations always exist**.

Proposition 5.2 characterizes the competitive equilibrium allocations of the competitive market with non-discriminatory demands as the dual allocations. We need Proposition 5.1 to prove it.

Proposition 5.1. *Let $(u,w;x)$ be a strongly-stable allocation. Then x is an optimal labor time allocation.*

Note that if x is an optimal labor time allocation and $(u,w;x')$ is a strongly stable allocation with $x' \neq x$, then x is not necessarily compatible with (u,w). This is because u and w are indexed according to matching x'.

Proposition 5.2. *The set of competitive equilibrium allocations of the competitive market with non-discriminatory demands coincides with the set of dual allocations.*

Corollary 5.3. *The set of dual allocations coincides with the set of non-discriminatory strongly-stable allocations .*

Proof. It is immediate from Proposition 5.2 plus Theorem 4.6. ■

We can now prove the existence theorem.

Theorem 5.4. *The set of competitive equilibrium allocations of the competitive market with non-discriminatory demands, the set of competitive equilibrium allocations of the competitive market with discriminatory demands, the set of stable allocations for the flexible market, the set of stable allocations for the rigid market and the core are non-empty.*

Proof. From the Duality Theorem we have that the dual allocations always exist. Let σ be a dual allocation. Corollary 5.3 implies that σ is strongly-stable, so it is setwise stable and so it is corewise-stable. Hence the result follows and the proof is complete.■

Remark 3.2.1 implies that the set of competitive equilibrium allocations for the competitive market with discriminatory demands contains the set of competitive equilibrium allocations for the competitive market with non-discriminatory demands. Proposition 5.2 then implies that the set of competitive equilibrium allocations for the competitive market with discriminatory demands contains the set of dual allocations. However, it is easy to construct examples in which this inclusion is strict.

6. FINAL REMARKS AND RELATED WORK

The present work introduced and analyzed, cooperatively and competitively, the time-sharing assignment game, aiming to contribute to a better understanding of the cooperative and competitive equilibrium concepts.

The cooperative analysis was conducted under the assumption of rigid agreements or flexible agreements. The associated competitive markets are of two kinds: with discriminatory demands and with non-discriminatory demands. The set of competitive equilibrium allocations of the competitive market with non-discriminatory demands is contained, and may be properly contained, in the set of competitive equilibrium allocations of the competitive market with discriminatory demands.

We put all these markets together and compared their corresponding cooperative and competitive equilibria. Such an innovation permited to distinguish five cooperative solution concepts: corewise-stability, setwise-stability, strong-stability, buyer-non-discriminatory strong-stability and non-discriminatory strong-stability. The definition of these concepts implies that the corresponding solution sets are set inclusion related: one is a super-set of the next. Examples showed that the set inclusion relation may be strict. These sets, as well as the two sets of competitive equilibrium allocations are the same under both kinds of agreements. A consequence of this is that these allocations can be identified by just using the characteristic function of the game. However, the characteristic function does not give enough information to identify the cooperative equilibria of the rigid market with the setwise-stable allocations and the cooperative equilibria of the flexible market with the strongly-stable allocations. The appropriate model was shown to be the deviation function form, introduced in Sotomayor (2011), which uses the concept of feasible deviation from a given allocation via a coalition. Intuitively, a feasible deviation consists of a set of actions allowed by the rules of the game which can be taken against the given allocation by the members of the coalition. These actions are not captured by the characteristic function of the time-sharing assignment game. The representation given by the deviation function form allowed to show that:

(A) The corewise-stability does not capture the idea of cooperative equilibrium in both cooperative models;

(B) the cooperative equilibrium allocations of the rigid model are identified with the setwise-stable allocations;

(C) the cooperative equilibrium allocations of the flexible model are identified with the strongly-stable allocations.

(D) the competitive equilibrium allocations of the competitive market with discriminatory demands are characterized as the stable allocations of the flexible model which do not discriminate the buyers;

(E) the competitive equilibrium allocations of the competitive market with non-discriminatory demands are characterized as the stable allocations of the flexible model which do not discriminate any agent;

(F) the stable allocations of the flexible model which do not discriminate any agent are identified with the dual allocations.

(B) and (C) and the definitions of setwise-stability and strong-stability then implied that the set of cooperative equilibrium allocations of the flexible model is contained in the set of cooperative equilibrium allocations of the rigid model. Example 2.2.2 illustrated that stable allocations under rigid agreements may be unstable under flexible agreements.

(D) and (E) established the connection between the cooperative and the competitive structures. Therefore, the competitive equilibrium allocations of the two competitive markets are stable in both cooperative markets, so they are in the core.

From the technical point of view, the set inclusion relation among the cooperative and competitive solution sets propitiated the proof of the existence theorem. The well-known results of Linear programming, the Duality theorem and the Linear programming equilibrium theorem, implied that the dual allocations always exist. The identification given in (F) was then used to prove the non-emptiness of the core, of the sets of stable allocations for the rigid and flexible markets and of the sets of competitive equilibrium allocations of both competitive markets.

The problem of correlating the cooperative e and competitive equilibrium concepts has been directly or indirectly treated in several matching models existent in the literature. In the assignment game, for instance, the set of competitive equilibrium allocations, the set of stable allocations and the core coincide and are non-empty (Shapley and Shubik, 1972). The same result applies in the many-to-one matching model of Kelso and Crawford (1982) and also in the many-to-many case with one-dimensional payoffs introduced in Sotomayor (1992). In the multiple-partners assignment game, these sets are non-empty and one is contained in the next, but they do not always coincide. Furthermore, the set of competitive equilibrium allocations under discriminatory demands is characterized as the set of stable allocations in which no seller discriminates any buyer and it is obtained by shrinking the set of stable allocations through an isotone function (Sotomayor 1992, 2007).

The technique of analysing the rigid and the flexilbe markets altogether possibilitated to compare the correlations between the cooperative and competitive structures obtained in the two markets. The novelty brought by the time-sharing assignment game is that these correlations are distinct from one another and differ from that one that was observed in the multiple-partners assignment game. The stable allocations of the rigid model in which the sellers do not discriminate the buyers are not necessarily competitive. This occurs in the flexible market. Nevertheless, the set of competitive equilibrium allocations are not obtained by shrinking the set of stable allocations for the flexible model through an isotone function.

It can be easily verified that all results of the present paper could be obtained if we required that the numbers $r(p)$'s, $s(q)$'s and x_{pq}'s were integers. The market of buyers and sellers of indivisible goods, in which the quota of a seller is the number of identical objects he owns, the quota of a buyer is the maximum number of objects he can acquire and a buyer is allowed to purchase more than one item from the same seller fits well in this model. Within this context, the multiple-partners assignment game is the restriction of the time-sharing assignment game to the case where each buyer can acquire one item at most from the same seller. This assumption causes the rigid and the flexible markets to coincide.

Another kind of connection between the cooperative and competitive structures of the rigid and flexible markets is considered in a companion paper. There, we obtain precise results by examining the algebraic structure of the five solution sets treated here.

Sotomayor (2010) extends the time-sharing assignment game with rigid agreements to a non-matching coalitional game, in which players form coalitions of any size. The concept of stability is identified with the appropriate version of the setwise-stability concept given here. It is proved there that the core may be bigger than the set of stable allocations.

A variation of the buyers and sellers market described above, where the objects of a seller may be distinct, is proposed by Jaume *et al* (2007). These authors concentrate their analysis on the algebraic structure of the set of competitive equilibrium price vectors, rather than on the algebraic structure of the set of competitive equilibrium allocations. Their competitive equilibrium concept is closely related to the competitive equilibrium concept for the market with non-discriminatory demands presented here. They prove that this set preserves the lattice structure that is observed in the previous models.

Camiña (2006) studies the particular case in which a unique seller owns all objects, non necessarily identical, and each buyer wants to buy one object at most. She shows that the set of core allocations is a non-empty complete lattice under the partial order defined by the preferences of the buyers and may be different from the set of competitive equilibrium allocations.

APENDIX: PROOFS OF PROPOSITIONS 4.1, 4.2 AND 5.2.

Proposition 4.1. *Let* $(u,w;x)$ *be a feasible allocation. Then* $(u,w;x)$ *is pairwise-strongly-stable if and only if for all* $p\in P$ *and feasible labor allocation* x' *we have that*

(*) $U_p \geq \sum_{q\in B(p,x')}(a_{pq}-w'_{pq})x'_{pq}$,

where $w'_{pq}x'_{pq} = w_{pq}x'_{pq}$ *if* $x_{pq}\geq x'_{pq}$, $w'_{pq}x'_{pq} = w_{pq}x_{pq} + w_{(p)q}(min)(x'_{pq}-x_{pq})$ *if* $0<x_{pq}<x'_{pq}$ *and* $w'_{pq}x'_{pq}= w_{(p)q}(min)x'_{pq}$ *if* $x'_{pq}>0$ *and* $x_{pq}=0$.

Proof. Suppose $(u,w;x)$ is pairwise-strongly-stable but there is some feasible labor allocation x' and $p\in P$ such that (*) is not satisfied. Then,

(1) $U_p < \sum_{q\in B(p,x')}(a_{pq}-w'_{pq})x'_{pq}$.

Define $I=\{q\in Q;\ x_{pq}\geq x'_{pq}>0\}$, $J=\{q\in Q;\ x_{pq}< x'_{pq}\}$ and $K=\{q\in Q;\ x_{pq}>0,\ x'_{pq}=0\}$.

Then, $\sum_{q\in I} u_{pq}x_{pq} + \sum_{q\in J\cap B(p,x)} u_{pq}x_{pq} + \sum_{q\in K} u_{pq}x_{pq} < \sum_{q\in I} (a_{pq}-w_{pq})x'_{pq} + \sum_{q\in J}(a_{pq}-w'_{pq})x'_{pq} = \sum_{q\in I} u_{pq}x'_{pq} + \sum_{q\in J\cap B(p,x)} [(a_{pq}-w_{pq})x_{pq} + (a_{pq}- w_{(p)q}(min))(x'_{pq}-x_{pq})] + \sum_{q\in J-B(p,x)} (a_{pq}- w_{(p)q}(min))x'_{pq} =\sum_{q\in I}$

$u_{pq}x'_{pq}$ + $\sum_{q\in J\cap B(p,x)}u_{pq}x_{pq}$ +$\sum_{q\in J}$ $(a_{pq}$- $w_{(p)q}(min))x'_{pq}$ -$\sum_{q\in J\cap B(p,x)}(a_{pq}$- $w_{(p)q}(min))x_{pq}$= $\sum_{q\in I}$ $u_{pq}x'_{pq}$+ $\sum_{q\in J\cap B(p,x)}u_{pq}x_{pq}+\sum_{q\in J}(a_{pq}- w_{(p)q}(min))(x'_{pq}-x_{pq})$, so

(2) $\sum_{q\in I} u_{pq}(x_{pq} - x'_{pq})$ + $\sum_{q\in K} u_{pq}x_{pq} < \sum_{q\in J} (a_{pq}- w_{(p)q}(min))(x'_{pq}-x_{pq})$.

Since $\sum_{q\in I}(x_{pq} - x'_{pq})$ = $(r(p)$ - $\sum_{q\in J\cup K} x_{pq})$ – $(r(p)$-$\sum_{q\in J} x'_{pq})$= $\sum_{q\in J} (x'_{pq}$ - $x_{pq})$ - $\sum_{q\in K} x_{pq}$ it follows that $u_{pr}\sum_{q\in J} (x'_{pq} - x_{pq})$ = $u_{pr} \sum_{q\in I}(x_{pq} - x'_{pq})$ +$u_{pr}\sum_{q\in K} x_{pq} \leq$ $\sum_{q\in I} u_{pq}(x_{pq} - x'_{pq})$ + $\sum_{q\in K} u_{pq}x_{pq}$, where $u_{pr}=min\{u_{pq};q \in I\cup K\}$.

By (2) we have

$u_{pr}\sum_{q\in J} (x'_{pq} - x_{pq}) < \sum_{q\in J} (a_{pq}- w_{(p)q}(min))(x'_{pq}-x_{pq})$.

Then,

$\sum_{q\in J}(a_{pq}- w_{(p)q}(min) - u_{pr})(x'_{pq}-x_{pq})>0$, so we must have $(a_{pq}- w_{(p)q}(min) - u_{pr}) >0$ for some $q \in J$. But $r\neq q$ because $r\in I\cup K$ and $q\in J$. Then $u_{pr} \geq$ $u_{p(q)}(min)$ and $0< (a_{pq}- w_{(p)q}(min) - u_{pr}) \leq a_{pq}$ - $w_{(p)q}(min)$ - $u_{p(q)}(min)$, so $u_{p(q)}(min) + w_{(p)q}(min)< a_{pq}$, which contradicts the assumption that $(u,w;x)$ is pairwise strongly-stable. Hence (*) is satisfied for all $p\in P$ and feasible labor allocation x'.

In the other direction, let (p,q^*) be an unsaturated pair (with respect to x). Then $x_{pq^*}<r(p)$ and $x_{pq^*}<s(q^*)$. Set $u_{pm} \equiv u_{p(q^*)}(min)$. Let λ be some positive number such that x_{pm}- $\lambda \geq 0$, $x_{pq^*} + \lambda\leq r(p)$ and $x_{pq^*} + \lambda\leq s(q^*)$. Consider a feasible labor time allocation x' such that $x'_{pq^*}= x_{pq^*} + \lambda$, $x'_{pm} =x_{pm}- \lambda$, $x'_{pk} = x_{pk}$ for all $k\notin\{q^*,m\}$. Then we have

$\sum_{k\in B(p,x)-\{q^*,m\}} u_{pk}x_{pk} + u_{pm}x_{pm} + u_{pq^*}x_{pq^*} =\sum_{k\in B(p,x)} u_{pk}x_{pk} = U_p \geq \sum_{k\in B(p,x)-\{q^*,m\}} (a_{pk} - w_{pk}) x_{pk}$ + $(a_{pm} - w_{pm})(x_{pm} - \lambda) + (a_{pq^*}-w_{pq^*})x_{pq^*} + (a_{pq^*}-w_{(p)q^*}(min))\lambda$, where the weak inequality follows from (*).

Then, $\lambda u_{pm} \geq (a_{pq^*}-w_{(p)q^*}(min))\lambda$ and hence $u_{pm} + w_{(p)q^*}(min) \geq a_{pq^*}$. Then, $u_{p(q^*)}(min)$ + $w_{(p)q^*}(min)\geq a_{pq^*}$, so $(u,w;x)$ is pairwise-strongly-stable and the proof is complete.■

Proposition 4.2. *Let $(u,w;x)$ be a feasible allocation. Then $(u,w;x)$ is strongly-stable if and only if it is pairwise-strongly-stable.*

Proof. Suppose $(u,w;x)$ is strongly-stable. If condition (p) did not hold for some unsaturated pair (p,q), then buyer p and seller q could increase their earnings by transferring part of their labor time from some other partnership to $\{p,q\}$ (which is possible since (p,q) is unsaturated) and both players could profit from the increased earnings so obtained, which is absurd.

In the other direction, suppose by contradiction that $(u,w;x)$ satisfies (p) but it is not strongly-stable. This means that $(u,w;x)$ must be strongly-quasi-dominated by a feasible allocation $(u^*,w^*;x^*)$ via some coalition $R\cup T$, with $R\subseteq P$ and $T\subseteq Q$. By Definition 2.2.5 – (i_1), we have that, for all $p\in R$ and for all $q\in T$

(1) $U_p < \sum_{q\in B(p,x^*)} u^*_{pq} x^*_{pq}$ and $W_q< \sum_{p\in B(q,x^*)} w^*_{pq} x^*_{pq}$.

Set $A\equiv\{(p,q) \in C(x^*);$ $p\in R$ or $q \in T\}$, $D\equiv\{(p,q) \in C(x^*);$ $p\in R$, $q \in T$ *and* $x_{pq}=0\}$ and $E\equiv\{(p,q) \in C(x^*);$ $p\in R$, $q \in T$ *and* $x_{pq}>0\}$.

Adding up (1) yields

(2) $\sum_{p\in R}U_p <\sum_{p\in R}\sum_{q\in B(p,x^*)} u^*_{pq} x^*_{pq}$ and $\sum_{q\in T} W_q < \sum_{q\in T} \sum_{p\in B(q,x^*)}w^*_{pq} x^*_{pq}$.

Definition 2.2.5 – (i_2) implies that $x_{pq} \geq x^*_{pq}$ and $u_{pq}=u^*_{pq}$ for all $(p,q)\in A$ with $q\notin T$ and $x_{pq} \geq x^*_{pq}$ and $w_{pq} =w^*_{pq}$ for all $(p,q)\in A$ with $p\notin R$. Then,

(3) $\sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)} u^*_{pq}\, x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)} w^*_{pq}\, x^*_{pq}$

$= \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq} x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq} x^*_{pq}$.

From (2) and (3) we get

$\sum_{p\in R} U_p + \sum_{q\in T} W_q < [\sum_{p\in R}\sum_{q\in T\cap B(p,x^*)} u^*_{pq}\, x^*_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*)} w^*_{pq}\, x^*_{pq}] + [\sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)} u^*_{pq}\, x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)} w^*_{pq}\, x^*_{pq}] = [(\sum_{p\in R}\sum_{q\in T\cap B(p,x^*) - B(p,x)} u^*_{pq}\, x^*_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*) - B(q,x)} w^*_{pq}\, x^*_{pq}) + (\sum_{p\in R}\sum_{q\in T\cap B(p,x^*)\cap B(p,x)} u^*_{pq}\, x^*_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w^*_{pq}\, x^*_{pq})] + [\sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq} x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq} x^*_{pq}]=$

(4) $\sum_{(p,q)\in D} (u^*_{pq} + w^*_{pq}) x^*_{pq} + \sum_{(p,q)\in E} (u^*_{pq} + w^*_{pq})\, x^*_{pq} + \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq} x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq} x^*_{pq}$.

The feasibility of $(u^*,w^*;x^*)$ implies that condition (e) is satisfied, so the expression in (4) is equal to

$\sum_{(p,q)\in D} a_{pq}\, x^*_{pq} + \sum_{(p,q)\in E} a_{pq}\, x^*_{pq} + \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq} x^*_{pq} +$
$\sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq} x^*_{pq}$.

From *(p)* and from the fact that all $(p,q)\in D$ are unsaturated, it follows that $\sum_{(p,q)\in D} a_{pq}\, x^*_{pq} \leq \sum_{(p,q)\in D} (u_{p(q)}(min) + w_{(p)q}(min))\, x^*_{pq}$. From definition of E we have that $\sum_{(p,q)\in E} a_{pq}\, x^*_{pq} = \sum_{(p,q)\in E} (u_{pq} + w_{pq})\, x^*_{pq}$. Then,

(5) $\sum_{p\in R} U_p + \sum_{q\in T} W_q < \sum_{(p,q)\in D} (u_{p(q)}(min) + w_{(p)q}(min))\, x^*_{pq} + \sum_{(p,q)\in E} (u_{pq} + w_{pq})\, x^*_{pq} + \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq} x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq} x^*_{pq}$.

We now have that for every $p\in R$,

$\sum_{q\in T\cap B(p,x^*)-B(p,x)} x^*_{pq} + \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} x^*_{pq} + \sum_{q\in B(p,x^*)\cap B(p,x)-T} x^*_{pq} = r(p) = \sum_{q\in B(p,x) - B(p,x^*)} x_{pq} + \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} x_{pq} + \sum_{q\in B(p,x)\cap B(p,x^*)-S} x_{pq}$, so

$\sum_{q\in T\cap B(p,x^*)-B(p,x)} x^*_{pq} = \sum_{q\in B(p,x) - B(p,x^*)} x_{pq} + \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} (x_{pq} - x^*_{pq}) + \sum_{q\in B(p,x^*)\cap B(p,x)-T} (x_{pq} - x^*_{pq})$. Using that $u_{p(q)}(min) = u_p(min)$ for all $q\in T-B(p,x)$ we have

(6) $\sum_{q\in T\cap B(p,x^*)-B(p,x)} u_{p(q)}(min)\, x^*_{pq} = \sum_{q\in T\cap B(p,x^*)-B(p,x)} u_p(min)\, x^*_{pq} = \sum_{q\in B(p,x) - B(p,x^*)} u_p(min)\, x_{pq} + \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_p(min)\, (x_{pq} - x^*_{pq}) + \sum_{q\in B(p,x^*)\cap B(p,x)-T} u_p(min)(x_{pq} - x^*_{pq}) \leq \sum_{q\in B(p,x) - B(p,x^*)} u_{pq}\, x_{pq} + \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq}\, (x_{pq} - x^*_{pq}) + \sum_{q\in B(p,x^*)\cap B(p,x)-T} u_{pq}\, (x_{pq} - x^*_{pq})$

Symmetrically, for every $q\in T$ we have that

(7) $\sum_{p\in R\cap B(q,x^*)-B(q,x)} w_{(p)q}(min) x^*_{pq} \leq \sum_{p\in B(q,x)-B(q,x^*)} w_{pq} x_{pq} + \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}(x_{pq} - x^*_{pq}) + \sum_{p\in B(q,x^*)\cap B(q,x)-R} w_{pq}(x_{pq} - x^*_{pq})$.

Adding up (6) and (7) yields

$\sum_{(p,q)\in D} (u_{p(q)}(min) + w_{(p)q}(min))\, x^*_{pq} \leq [\sum_{p\in R}\sum_{q\in B(p,x) - B(p,x^*)} u_{pq}\, x_{pq} + \sum_{p\in R} \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq}\, (x_{pq} - x^*_{pq}) + \sum_{p\in R} \sum_{q\in B(p,x^*)\cap B(p,x)-T} u_{pq}\, (x_{pq} - x^*_{pq})] + [\sum_{q\in T} \sum_{p\in B(q,x)-B(q,x^*)} w_{pq} x_{pq}) + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}(x_{pq} - x^*_{pq}) + \sum_{q\in T} \sum_{p\in B(q,x^*)\cap B(q,x)-R} w_{pq}(x_{pq} - x^*_{pq})] = [(\sum_{p\in R}\sum_{q\in B(p,x) - B(p,x^*)} u_{pq}\, x_{pq} + \sum_{p\in R} \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq}\, x_{pq} + \sum_{p\in R} \sum_{q\in B(p,x^*)\cap B(p,x)-S} u_{pq}\, x_{pq})] + [(\sum_{q\in T} \sum_{p\in B(q,x)-B(q,x^*)} w_{pq} x_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq} x_{pq} + \sum_{q\in T} \sum_{p\in B(q,x^*)\cap B(q,x)-R} w_{pq} x_{pq})] - [\sum_{p\in R} \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq} x^*_{pq} + \sum_{p\in R} \sum_{q\in B(p,x^*)\cap B(p,x)-T} u_{pq} x^*_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}\, x^*_{pq} + \sum_{q\in T} \sum_{p\in B(q,x^*)\cap B(q,x)-R} w_{pq} x^*_{pq}] = \sum_{p\in R} U_p + \sum_{q\in T} W_q - [\sum_{p\in R} \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq}\, x^*_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}\, x^*_{pq}] - [\sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap}$

$_{B(q,x)}u_{pq}x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq}x^*_{pq}] = \sum_{p\in R}U_p + \sum_{q\in T} W_q - \sum_{(p,q)\in E} (u_{pq} + w_{pq})\, x^*_{pq} - \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq}x^*_{pq} - \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq}x^*_{pq}$. Then,

$\sum_{(p,q)\in D} (u_{p(q)}(min)+w_{(p)q}(min))\; x^*_{pq} \leq \sum_{p\in R}U_p + \sum_{q\in T} W_q - \sum_{(p,q)\in E} (u_{pq} + w_{pq})\; x^*_{pq} - \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq}x^*_{pq} - \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq}x^*_{pq}$, so

$\sum_{p\in R}U_p + \sum_{q\in T} W_q \geq \sum_{(p,q)\in D} (u_{p(q)}(min)+w_{(p)q}(min))\; x^*_{pq} + \sum_{(p,q)\in E} (u_{pq} + w_{pq})\; x^*_{pq} + \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq}x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq}x^*_{pq}$, which contradicts (5). Hence *(u,w;x)* is strongly-stable. ■

For simplicity of notation, in what follows, we will use some times $\sum_P$, $\sum_Q$, $\sum_{PxQ}$ to denote, respectively, $\sum_{p\in P}$, $\sum_{q\in Q}$, $\sum_{(p,q)\in PxQ}$ and so on.

Proposition 5.1. *Let (u,w;x) be a strongly-stable allocation. Then x is an optimal labor time allocation.*

Proof. Let x' be any feasible labor time allocation. For all $(p,q)\in P\times Q$, let $\Delta_{pq} = x_{pq} - x'_{pq}$. We must show

(1) $\sum_{PxQ} a_{pq}\, \Delta_{pq} \geq 0$.

Define $T\equiv\{(p,q)\in C(x);\; x_{pq}-x'_{pq}\geq 0\}$, $T^*\equiv\{(p,q)\in C(x);\; x_{pq}-x'_{pq} < 0\}$, $T(p)\equiv\{q;\; (p,q)\in T\}$, $T^*(p)\equiv\{q;\; (p,q)\in T^*\}$, $T(q)\equiv\{p;\; (p,q)\in T\}$ and $T^*(q)\equiv\{p;\; (p,q)\in T^*\}$. Then,

(2) $\sum_{q\in T(p)} \Delta_{pq} + \sum_{q\in T^*(p)} \Delta_{pq} = 0$ for all p and $\sum_{p\in T(q)} \Delta_{pq} + \sum_{p\in T^*(q)} \Delta_{pq} = 0$ for all q, by feasibility of x and x'. Set

(3) $u_p \equiv min\{ u_{pq};\, q\in T(p)\}$ and $w_q \equiv min\{ w_{pq};\, p\in T(q)\}$.

Then, $\sum_{C(x)} a_{pq}\, \Delta_{pq} = \sum_T a_{pq}\, \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} = \sum_T (u_{pq} + w_{pq})\, \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq}$
$= \sum_P \sum_{q\in T(p)} u_{pq}\, \Delta_{pq} + \sum_Q \sum_{p\in T(q)} w_{pq}\, \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} \geq \sum_P u_p \sum_{q\in T(p)} \Delta_{pq} +$
$+\sum_Q w_q \sum_{p\in T(q)} \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} = -\sum_P u_p \sum_{q\in T^*(p)} \Delta_{pq} - \sum_Q w_q \sum_{p\in T^*(q)} \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} = \sum_{T^*} -(u_p + w_q)\, \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} = \sum_{T^*} (a_{pq} - (u_p + w_q))\, \Delta_{pq}$, where the third last equality follows from (2).

Since $(p,q)\in T^*$ we have $x_{pq} < x'_{pq}$, so $x_{pq} < r(p)$ and $x_{pq} < s(q)$ and then (p,q) is unsaturated. From (3) $u_p = u_{pm}$ for some $m\in T(p)$ and $w_q = w_{kq}$ for some $k\in T(q)$, so $m\neq q$ and $k\neq p$ because $(p,q)\in T^*$, so $u_p \geq u_{p(q)}(min)$ and $w_q \geq w_{(p)q}(min)$ for all $(p,q)\in T^*$. Finally, by strong stability, $(a_{pq} - (u_{p(q)}(min) + w_{(p)q}(min))) \leq 0$ for all $(p, q) \in T^*$. We also have $\Delta_{pq} < 0$ for all $(p, q) \in T^*$, so $\sum_{T^*} (a_{pq} - (u_p + w_q))\, \Delta_{pq} \geq 0$ and so (1) is proved. ■

Proposition 5.2. *The set of competitive equilibrium allocations of the competitive market with non-discriminatory demands coincides with the set of dual allocations.*

Proof. Let *(u,w;x)* be a dual allocation. It is implied by (D) and by the construction of *(u,w;x)* that this allocation is feasible. Property *(p)* is implied by (C) and (D); Theorem 4.3 implies that *(u,w;x)* is strongly-stable. Now use the definition of *u* and *w* to get that *(u,w;x)* is non-discriminatory and then, by Theorem 4.6, it is a competitive equilibrium allocation of the competitive market with non-discriminatory demands.

Conversely, let $(u,w;x)$ be a competitive equilibrium allocation of the competitive market with non-discriminatory demands. Define (y,z) such that $y_p=u_p(min)$ and $z_q=w_q(min)$ for all $p\in P$ and $q\in Q$. Theorem 4.6 implies that $(u,w;x)$ is strongly-stable and

(1) $u_{pq}=u_p(\min)=y_p$ and $w_{pq}=w_q(\min)=z_q$ for all $(p,q)\in P\times Q$.

Proposition 5.1 implies that x is an optimal labor time allocation, so it is an optimal solution of (P1). Theorem 4.3 implies that (p) is satisfied, so $y_p+z_q\geq a_{pq}$ if $x_{pq}=0$. By (1) it follows that $y_p + z_q=a_{pq}$ if $x_{pq}>0$, so (B2) is satisfied. The feasibility of $(u,w;x)$ implies that (y,z) minimizes (B1) and (B3) is satisfied. Hence $(u,w;x)$ is a dual allocation and the proof is complete. ■

REFERENCES

[1] Camiña, E. (2006) "A generalized assignment game", *Mathematical Social Sciences* 52, 152-161.

[2] Fagebaume. A., D. Gale and M. Sotomayor (2010) "A note on the multiple partners assignment game", *Journal of Math. Economics* 46, 388-392.

[3] Gale, D. (1960) *The Theory of Linear Economic Models*. New York: McGraw Hill.

[4] Gale, D. and L. Shapley (1962) "College admissions and the stability of marriage", *American Mathematical Monthly,* 69, 9-15.

[5] Gul, F. and E. Stacchetti (1999) "Walrasian equilibrium with gross substitutes", *Journal of Economic Theory* 87, 95-124.

[6]________________(2000) "The English auction with differentiated commodities", *Journal of Econ. Theory* 92, 66–95.

[7] Jaume, D., J. Massó and A. Neme (2007) "The multiple-partners assignment game with heterogeneous sell and multi-unit demands: Competitive equilibria", *mimeo.*

[8] Kelso, A. and V. P. Crawford (1982) "Job matching, coalition formation, and gross substitutes", *Econometrica* 50, 1483-1504.

[9] Roth, A. and M. Sotomayor (1990) "Two-sided matching: a study in game theoretic modeling and analysis", Cambridge University Press.

[10] Shapley, L. and M. Shubik (1972) "The assignment game I: The core", *International Journal of Game Theory* 1, 111-130.

[11] Sotomayor, M. (1992) "The multiple partners game", *Dynamics and Equilibrium; Essays in honor to D. Gale,* edit. by M. Majundar, published by Mcmillian, 322 - 336.

[12] __________(1999-a) "The lattice structure of the set of stable outcomes of the multiple partners assignment game", *International Journal of Game Theory* 28: 567-583.

[13] __________(1999-b) "Three remarks on the many-to-many stable matching problem" *Mathematical Social Sciences* 38 (1999) 55–70

[14] __________(2002) "A labor market with heterogeneous firms and workers", *International Journal of Game Theory* 31, 269-283.

[15] __________ (2007) "Connecting the cooperative and competitive structures of the multiple-partners assignment game", *Journal of Economic Theory* 134: 155-174.

[16]________________(2009)"Adjusting prices in the multiple-partners assignment game", *Int J Game Theory* 38:575–600

[17] __________ (2010) "Stability property of matchings is a natural solution concept in coalitional market games", *International Journal of Game Theory* 39, 237-248

[18] ___________(2011) "Modeling cooperative decision situations: the deviation function form and the concept of stability", *mimeo.*

[19] Thompson, G. L. (1980) "Computing the core of a market game". *Extremal Methods and Systems Analysis,* AV Fiacco and KO Kortanek (eds). Lecture Notes in Economics and Mathematical Systems no. 174, Berlin: Springer, pp. 312-334.