TWO FOLK MANIPULABILITY THEOREMS IN THE GENERAL ONE-TO-ONE TWO-SIDED MATCHING MARKETS WITH MONEY

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Abstract:

We prove a "General Manipulability Theorem" for general one-to-one two-sided matching markets with money. This theorem implies two folk theorems, the Manipulability Theorem and the General Impossibility Theorem, and provides a sort of converse of the Non-Manipulability Theorem (Demange, 1982, Leonard, 1983, Demange and Gale, 1985).

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TWO FOLK MANIPULABILITY THEOREMS IN THE GENERAL ONE-TO-ONE TWO-SIDED MATCHING MARKETS WITH MONEY[1]

By

David Pérez-Castrillo[2] and Marilda Sotomayor[3]

ABSTRACT

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INTRODUCTION

We study two-sided matching markets where a finite number of heterogeneous sellers (or workers) meet a finite number of buyers (or firms). Each seller is willing to sell one object, and each buyer wants to buy, at most, one object. Both types of agents derive utility from money; however, their utility may also depend on the identity of their partner. That is, we consider situations where objects (i.e., sellers) are different from a buyer's point of view and where a seller may have preferences over buyers.

The general two-sided matching model with money that we analyze was proposed by Demange and Gale (1985). The authors generalized the assignment game, introduced in Shapley and Shubik (1972), in which the utility functions in money are linear and each seller is only concerned about the reward he may receive from his object; that is, he is indifferent as to the identity of the buyer. For ease of exposition, the main body of the paper will focus on the assignment game; we will extend the results at the end of the paper, in Section 5.

These matching models can be viewed as markets (competitive markets) operating as an exchange economy. We refer to each of the assignment games as a *buyer-seller market*. In such a market, given a vector of prices, each seller is ready to sell his object if the price exceeds his valuation, and each buyer demands the set of objects that maximize her surplus. Roughly speaking, a competitive equilibrium is a vector of prices, one for each object, and an allocation of (some of the) objects to buyers such that the demand of every buyer is satisfied, the price of every unsold object is its reservation price, and no two buyers obtain the same object. The set of equilibrium prices is non-empty (Gale, 1960) and is a complete lattice whose extreme points are the minimum and the maximum equilibrium prices (Shapley and Shubik, 1972), which are called buyer-optimal and seller-optimal equilibrium prices, respectively.[4, 5]

A competitive equilibrium mechanism is a function that selects a specific competitive equilibrium allocation for every buyer-seller market. Given a set of buyers

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[4] This competitive market was proposed in Gale (1960). Demange, Gale and Sotomayor (1986) introduced the term "minimum competitive prices", also called "minimum equilibrium prices" in Roth and Sotomayor (1990). The concept of a competitive equilibrium for the many-to-many matching model with additively separable utilities was introduced in Sotomayor (2007).

[5] The results stated and proved in this paper are established for the competitive market game; however, these results can easily be translated to the cooperative model because, in this model, the core coincides with the set of competitive equilibrium payoffs (Shapley and Shubik, 1972).

and sellers, the restriction of a competitive equilibrium mechanism to the set formed by all buyer-seller markets that keep unchanged the given set of agents defines a *competitive equilibrium rule associated to the mechanism*. That is, for every such market, the rule selects a specific competitive equilibrium for the corresponding market.

When a competitive equilibrium rule is adopted for use in a particular buyer-seller market, information about the valuation of the agents is required. Therefore, the rule induces a strategic game in which the set of players is the given set of agents; the preferences of these agents are derived from their true valuations; the strategies of a player are all possible valuations that s/he can report; and the outcome function is defined by the competitive equilibrium rule.

This paper studies the agents' incentives to report truthfully if a competitive equilibrium rule is used. The first important result in this respect is the *Non-Manipulability Theorem*, first proposed by Demange (1982) and Leonard (1983) and generalized by Demange and Gale (1985). These authors proved that if the buyer-optimal (or seller-optimal) competitive equilibrium rule is used, then in the induced strategic game, truth telling is a dominant strategy for each buyer (seller).[6] The Non-Manipulability Theorem implies that the mechanism that yields the optimal competitive equilibrium for a given side of the market is non-manipulable (i.e., it is strategy-proof) by the agents on that side.

However, if we consider the strategic incentives of the agents of both sides of the market, the previous positive result does not hold. The *Impossibility Theorem* (Theorem 7.3 of Roth and Sotomayor, 1990) asserts that any competitive equilibrium mechanism for the class of buyer-seller markets is manipulable. The proof of the theorem consists in finding a market such that, under any competitive equilibrium mechanism, there is an agent who has an incentive to misrepresent her/his valuation. Additionally, Demange and Gale (1985) present several examples of markets where a competitive equilibrium rule that yields the optimal competitive equilibrium for a given side of the market provides incentives to an agent belonging to the other side to increase her/his payoff by misrepresenting her/his valuation. The main feature of all these examples is that the markets have more than one vector of equilibrium prices (if there is only one vector of equilibrium prices, then the Non-Manipulability Theorem implies that no agent has

[6] Demange and Gale (1985) extended the non-manipulability theorem to the general two-sided matching model with money and to any competitive equilibrium rule that maps the market defined by the true valuations to the buyer-optimal (or seller-optimal) competitive equilibrium.

incentives to misrepresenting her/his valuation).

It has been believed that the phenomena observed in the particular market used in the proof of the Impossibility Theorem and in the examples of Demange and Gale (1985) happen in any market with more than one vector of equilibrium prices. This belief has supported, along the years, the following *folk theorems*, which have never been proved in the literature:

"**Folk Manipulability Theorem.** *Consider the buyer-optimal (respectively, seller-optimal) competitive equilibrium rule. Suppose that the market defined by the true valuations of the agents has more than one vector of equilibrium prices. If the buyer-optimal (respectively, seller-optimal) competitive equilibrium rule is used then, there is a seller (buyer) who can profitably misrepresent his (her) valuations, assuming the other agents tell the truth.*"

Folk General Impossibility Theorem. *Suppose that the market defined by the true valuations of the agents has more than one vector of equilibrium prices. No competitive equilibrium rule exists for which, in the game induced, truth telling is a dominant strategy for every agent.*

In the present work, we provide the proofs and formal statements of the two folk theorems, aiming to fill this gap in the literature. Indeed, we prove a stronger and more general *Manipulability Theorem* than the result suggested by the examples of Demange and Gale (1985)**,** because our theorem does not require the competitive equilibrium rule to produce the optimal competitive equilibrium for one of the sides. Our result has the two folk theorems as immediate corollaries.

General Manipulability Theorem. *Consider any competitive equilibrium rule. Suppose that the market M defined by the true valuations of the agents has more than one vector of equilibrium prices. If the rule applied to M does not yield a buyer-optimal (respectively, seller-optimal) competitive equilibrium for M, then any buyer (respectively, seller) who is not receiving his (her) optimal competitive equilibrium payoff for M can profitably misrepresent his (her) valuations, assuming the others tell the truth.*

The General Manipulability Theorem provides a sort of converse of the Non-Manipulability Theorem of Demange and Gale (1985). It implies that a competitive equilibrium rule is strategy-proof for the buyers (sellers) if and only if the rule maps the profile of true valuations to the buyer-optimal (seller-optimal) competitive equilibrium price. This result is mathematically unusual because it provides a way of concluding that a competitive equilibrium rule is, for example, strategy-proof for the buyers based only on the direct examination of the agents' payoffs obtained by the profile of true valuations.

Finally, the General Manipulability Theorem implies that there is no competitive equilibrium rule for a buyer-seller market with more than one equilibrium price vector, for which a truthful report of valuations for buyers and sellers is a Nash equilibrium. It also implies that no competitive equilibrium rule is strategy-proof.

This note is organized as follows. In Section 2 we present the framework and certain known results that will be used in Section 4. The competitive equilibrium mechanisms and the competitive equilibrium rules are introduced in Section 3. In Section 4 we state and prove our main results. The extension to the general case and the concluding remarks are presented in Section 5.

2-THE FRAMEWORK AND PRELIMINARIES

There are two non-empty, finite and disjoint sets of agents, P and Q, which can be thought of as being buyers and sellers, respectively. The set P has m buyers, $P = \{p_1, p_2,\ldots, p_m\}$, and the set Q has n sellers, $Q = \{q_1, q_2,\ldots, q_n\}$. Each seller q_k owns one indivisible object, and each buyer p_j wants to buy at most one of those objects. We use the same notation for a seller and for his object and reserve letters j and k to index buyers and sellers (or objects), respectively. Seller q_k assigns a value of $s_k \geq 0$ to his object. Buyer p_j assigns a value of $\alpha_{jk} \geq 0$ to object q_k. Thus, if buyer p_j purchases object q_k at price $\rho_k \geq s_k$, then her payoff is $u_j = \alpha_{jk} - \rho_k$ and the payoff of seller q_k is $v_k = \rho_k - s_k$. We denote by α_j the vector of the values of α_{jk}'s; the valuation matrix of the buyers and the valuation vector of the sellers are denoted by α and s, respectively. The model we have just described will be called a buyer-seller market and will be denoted by $M(P, Q; \alpha, s)$. If each seller's reservation price is 0, the corresponding model is the well-known Assignment Game presented in Roth and Sotomayor (1990), first proposed by Shapley and Shubik (1972).

An object q_k is *acceptable* to buyer p_j if and only if the potential gain from a trade between p_j and q_k is non-negative, that is, $\alpha_{jk} - s_k \geq 0$. Thus, an object is not acceptable to a buyer if there is no price at which the buyer wishes to buy and the seller wishes to sell the object. We denote $a(\alpha, s)_{jk} \equiv \alpha_{jk} - s_k$ if $\alpha_{jk} - s_k \geq 0$ and $a(\alpha, s)_{jk} \equiv 0$ otherwise. Finally, we use the notation $\sum_j$ for the sum over all p_j's in P, $\sum_k$ for the sum over all q_k's in Q, and $\sum_{j,k}$ for the sum over all p_j's in P and all q_k's in Q.

> **Definition 1.** *A* ***matching*** *for market $M(P, Q; \alpha, s)$ is identified with a matrix $x = (x_{jk})$ of zeros and ones. A matching x for $M(P, Q; \alpha, s)$ is* ***feasible*** *if it satisfies (a) $\sum_j x_{jk} \leq 1$ for all $q_k \in Q$, (b) $\sum_k x_{jk} \leq 1$ for all $p_j \in P$ and (c) $\alpha_{jk} - s_k \geq 0$ if $x_{jk} = 1$.*

Conditions (a) and (b) in Definition 1 state that a feasible matching assigns each object to at most one buyer and each buyer to at most one object. Condition (c) requires that the object assigned to a buyer by the feasible matching must be acceptable to her.

If $x_{jk} = 0$ for all $q_k \in Q$ (respectively, $p_j \in P$), we say that buyer p_j (respectively, seller q_k) is *unmatched.* If $x_{jk} = 1$, we say that buyer p_j is matched to seller q_k or that q_k is matched to p_j.

> **Definition 2.** *A* ***matching*** *x for market $M(P, Q; \alpha, s)$ is* ***optimal*** *if (a) it is feasible and (b) $\sum_{j,k} a(\alpha, s)_{jk}\, x_{jk} \geq \sum_{j,k} a(\alpha, s)_{jk}\, x'_{jk}$ for all feasible matching x'.*

A *feasible price vector* (or a *vector of feasible prices*) ρ for market $M(P, Q; \alpha, s)$ is a function from Q to R (the set of real numbers) such that $\rho_k \equiv \rho(q_k)$ is greater than or equal to s_k. A *feasible allocation* for $M(P, Q; \alpha, s)$ is a pair (ρ, x), where ρ is a feasible price vector and x is a feasible matching. The payoff of buyer $p_j \in P$ corresponding to a feasible allocation (ρ, x) is feasibly defined as $u_j = \alpha_{jk} - \rho_k$ if $x_{jk} = 1$ and $u_j = 0$ if p_j is unmatched under x. The payoff of seller $q_k \in Q$ is defined as $v_k = \rho_k - s_k$ if $x_{jk} = 1$ for some $p_j \in P$ and $v_k = 0$ if q_k is unmatched at x.

The *demand set* of buyer p_j at the feasible prices ρ is the following:

$D(p_j, \rho) = \{q_k \in Q;\ \alpha_{jk} - \rho_k \geq 0$ and $\alpha_{jk} - \rho_k \geq \alpha_{jt} - \rho_t$ for all $q_t \in Q\}$.

Observe that if $q_k \in D(p_j, \rho)$ then $\alpha_{jk} - s_k \geq 0$, so q_k is acceptable to p_j. However, we may have that q_k is acceptable to p_j but $\alpha_{jk} - \rho_k < 0$ or $\alpha_{jk} - \rho_k < \alpha_{jt} - \rho_t$ for some $q_t \in Q$. Thus, among all acceptable objects for buyer p_j that give her a non-negative payoff, the demand set of p_j only includes those objects that maximize her payoff given the prices.

Definition 3. *A feasible allocation* (ρ, x) *for* $M(P, Q; \alpha, s)$ *is a* ***competitive equilibrium*** *if (a) for all pairs* (p_j, q_k) *with* $x_{jk} = 1$, *we have* $q_k \in D(p_j, \rho)$; *(b) if* p_j *is unmatched, then* $\alpha_{jk} - \rho_k \leq 0$ *for all* $q_k \in Q$ *and (c) if* q_k *is unmatched, then* $\rho_k = s_k$.

If (ρ, x) is a competitive equilibrium for $M(P, Q; \alpha, s)$, ρ is called *equilibrium price vector* or simply *equilibrium prices* and x is called *competitive matching.* In this case, we say that x is *compatible* with ρ and vice-versa. The corresponding payoff vector (u, v), defined above, is called *competitive equilibrium payoff*, and $(u, v; x)$ is called *competitive equilibrium outcome*. We say that x is compatible with (u, v) if $(u, v; x)$ is a competitive equilibrium outcome.

The following well-known results will be used in the next sections.

Proposition A (Shapley and Shubik, 1972). (*a) If* x *is an optimal matching for market* $M(P, Q; \alpha, s)$, *then* x *is compatible with any competitive equilibrium payoff* (u, v) *for* $M(P, Q; \alpha, s)$; *(b) if* $(u, v; x)$ *is a competitive equilibrium outcome for* $M(P, Q; \alpha, s)$, *then* x *is an optimal matching for* $M(P, Q; \alpha, s)$.

Proposition A implies that if x is an optimal matching and ρ is a vector of equilibrium prices for $M(P, Q; \alpha, s)$, then (ρ, x) is a competitive equilibrium for $M(P, Q; \alpha, s)$. Conversely, if (ρ, x) is a competitive equilibrium for $M(P, Q; \alpha, s)$, then x is an optimal matching for $M(P, Q; \alpha, s)$.

Proposition B (Demange and Gale, 1985). *Let* (u, v) *be a competitive* *equilibrium payoff for* $M(P, Q; \alpha, s)$*. Then, if* $u_j > 0$ *(respectively,* $v_k > 0$*),* p_j *(respectively,* q_k*) is matched at any optimal matching for* $M(P, Q; \alpha, s)$*.*

Finally, we introduce two particularly interesting competitive equilibria.

Definition 4. *The competitive equilibrium payoff* $(\overline{u}, \underline{v})$ *is the* ***P-optimal competitive equilibrium payoff*** *if* $\overline{u}_j \geq u_j$ *for all* $p_j \in P$ *and for all competitive equilibrium payoffs* (u, v)*. The* ***Q-optimal competitive equilibrium payoff*** $(\underline{u}, \overline{v})$ *is symmetrically defined.*

Shapley and Shubik (1972) proved that in $M(P, Q; \alpha, s)$ the P-optimal competitive equilibrium payoff $(\overline{u}, \underline{v})$ corresponds to the *minimum equilibrium prices* $\underline{\rho} \equiv \underline{v} + s$. The equilibrium price vector $\underline{\rho}$ is at least as small in every component as any other equilibrium price vector. If x is an optimal matching, then $(\underline{\rho}, x)$ is called a *P-optimal competitive equilibrium*. Symmetrically, the *Q-optimal competitive equilibrium payoff* $(\underline{u}, \overline{v})$ corresponds to the *maximum competitive equilibrium prices* $\overline{\rho} \equiv \overline{v} + s$, and $(\overline{\rho}, x)$ is called a *Q-optimal competitive equilibrium*.

3. COMPETITIVE EQUILIBRIUM MECHANISMS AND COMPETITIVE EQUILIBRIUM RULES

A *competitive equilibrium mechanism* for the buyer-seller market is a function that selects a specific competitive equilibrium allocation for every market, that is, for every possible set of buyers and sellers, for every possible valuation matrix for the buyers and for every possible valuation vector for the sellers.

In what follows, we consider a fixed set P of buyers and a fixed set Q of sellers. For all (α, s), we will set $M(\alpha, s) \equiv M(P, Q; \alpha, s)$. A competitive equilibrium mechanism can be used to define an *associated competitive equilibrium rule (competitive equilibrium rule,* for short) (Π, X) where Π is a price rule and X is a matching rule. The domain of (Π, X) is the set of all markets $M(\alpha, s)$. Since P and Q are fixed, the value of (Π, X) for $M(\alpha, s)$ will be denoted simply by $(\Pi(\alpha, s), X(\alpha, s))$. Then, $(\Pi(\alpha, s), X(\alpha, s))$ is the competitive equilibrium selected by the mechanism when it is applied to market $M(\alpha, s)$. The corresponding buyers' payoff vector is denoted by $u(\alpha, s)$.

If the rule (Π, X) produces the P-optimal (respectively, Q-optimal) competitive equilibrium for every $M(\alpha, s)$, it is called the P-optimal (respectively, Q-optimal) competitive equilibrium rule.

When a competitive equilibrium rule (Π, X) is adopted for use in a particular market $M(\alpha, s)$ and the agents are asked to report their valuations, then the rule induces a strategic game $\Gamma(\Pi, X)$, where the set of players is the set of agents $P \cup Q$; the set of strategies for buyer p_j is the set of vectors $\alpha'_j \in \mathrm{R}^m_+$; the set of strategies for seller q_k is the set of numbers $s'_k \geq 0$; and the outcome function is given by the rule. Thus, if the agents report the strategies (α', s'), the outcome function produces $(\Pi(\alpha', s'), X(\alpha', s'))$, which is the value of the rule applied to the market $M(\alpha', s')$. The preferences of the players over the allocations that can be produced by (Π, X) are given by their preferences over the corresponding true payoffs, which are determined by their true valuations (α, s).[7] Thus, the true payoffs of buyer p_j and seller q_k with respect to market $M(\alpha, s)$ at allocation $(\Pi(\alpha', s'), X(\alpha', s'))$ are as follows:

$U_j(\Pi(\alpha', s'), X(\alpha', s'); \alpha, s) = \alpha_{jk} - \Pi_k(\alpha', s')$ if $X(\alpha', s')_{jk} = 1$,

$U_j(\Pi(\alpha', s'), X(\alpha', s'); \alpha, s) = 0$, if p_j is unmatched at $X(\alpha', s')$,

$V_k(\Pi(\alpha', s'), X(\alpha', s'); \alpha, s) = \Pi_k(\alpha', s') - s_k$ if q_k is matched at $X(\alpha', s')$ and

$V_k(\Pi(\alpha', s'), X(\alpha', s'); \alpha, s) = 0$ if q_k is unmatched at $X(\alpha', s')$.

When (α, s) is the profile of true valuations for the agents, we will sometimes refer to the market $M(\alpha, s)$ as the *true market*.

Definition 5. *A competitive equilibrium rule (Π, X) is* ***manipulable via $M(\alpha, s)$*** *if there is an agent $y_i \in P \cup Q$ and a profile of valuations (α', s'), which differs from (α, s) only in the y's valuations, such that $U_i(\Pi(\alpha', s'), X(\alpha', s'); \alpha, s) > U_i(\Pi(\alpha, s), X(\alpha, s); \alpha, s)$, if $y_i \in P$ and $V_i(\Pi(\alpha', s'), X(\alpha', s'); \alpha, s) > V_i(\Pi(\alpha, s), X(\alpha, s); \alpha, s)$ if $y_i \in Q$. In this case, we say that agent y_i*

[7] We will use "primes" to denote reported variables. For example, α is the true valuation matrix of the buyers, whereas α' is the reported valuation matrix of the buyers. The domain of the outcome function is the set of all possible reports.

manipulates the rule (Π, X) via M(α, s).

A rule is **non-manipulable** if, for every market $M(\alpha, s)$, there is no agent y who can manipulate the rule via $M(\alpha, s)$. If a rule is non-manipulable it is called **strategy-proof.** A competitive equilibrium mechanism is **manipulable** some associate competitive equilibrium rule is manipulable via some market.

4. MAIN RESULTS

In this section, we prove the two folk theorems stated in Section 1. Both theorems are corollaries of a stronger and more general result that we call the General Manipulability Theorem. In these results, we consider competitive equilibrium rules and analyze the agents' equilibrium behavior when the true valuations of buyers and sellers determine a buyer-seller market with more than one competitive equilibrium price vector.

Theorem 1. (**General Manipulability Theorem**) *Let (Π, X) be any competitive equilibrium rule. Let $M \equiv M(\alpha, s)$ be a market with more than one competitive equilibrium price vector. For $Y \in \{P, Q\}$, suppose that $(\Pi(\alpha, s); X(\alpha, s))$ is not a Y-optimal competitive equilibrium for M. Then, any $y \in Y$ whose payoff at $(\Pi(\alpha, s); X(\alpha, s))$ is different from her/his payoff under the Y-optimal competitive equilibrium for M can manipulate (Π, X) via M.*

Proof. The competitive equilibrium outcome if agents select (α, s) is $(u(\alpha, s), \Pi(\alpha, s) - s; X(\alpha, s))$, where $u(\alpha, s)$ is the corresponding payoff vector for the buyers feasibly defined. Let $(\overline{u}, \underline{\rho} - s)$ and $(\underline{u}, \overline{\rho} - s)$ be the buyer-optimal and the seller-optimal competitive equilibrium payoffs for M. Proposition A implies that $X(\alpha, s)$ is compatible with $(\overline{u}, \underline{\rho} - s)$ and $(\underline{u}, \overline{\rho} - s)$.

First case: $Y = P$. By hypothesis, $u(\alpha, s) \neq \overline{u}$. Let p_j be any buyer such that $\overline{u}_j > u(\alpha, s)_j \geq 0$. Given that $\overline{u}_j > 0$, Proposition B implies that p_j is matched to some q_k at $X(\alpha, s)$, so $\alpha_{jk} - \overline{u}_j = \underline{\rho}_k$. Furthermore, for some positive λ, $\overline{u}_j > \overline{u}_j - \lambda > u(\alpha, s)$.

Now define α' as follows: $\alpha'_{it} = \alpha_{it}$ for all $(p_i, q_t) \in PxQ$ with $p_i \neq p_j$; $\alpha'_{jk} =$

$\alpha_{jk} - (\bar{u}_j - \lambda)$, and $\alpha'_{jt} = 0$ for all $q_t \in Q$ with $q_t \neq q_k$. We have $\alpha_{jk} - (\bar{u}_j - \lambda) > \alpha_{jk} - \bar{u}_j = \underline{\rho}_k \geq 0$, which implies $\alpha'_{jk} \geq 0$; hence, α' is well defined.

We are going to show that $U_j(\Pi(\alpha', s), X(\alpha', s); \alpha, s) > u_j(\alpha, s)$. In fact, note that $(\underline{\rho}, X(\alpha, s))$ is a competitive equilibrium for $M(\alpha', s)$ because p_j still wants to demand q_k and the demand sets of the other buyers do not change. Furthermore, in market $M(\alpha', s)$, $D(p_j, \underline{\rho}) = \{q_k\}$. This fact implies that p_j is matched to q_k at any competitive matching for $M(\alpha', s)$. Therefore, $U_j(\Pi(\alpha', s), X(\alpha', s); \alpha, s) = \alpha_{jk} - \Pi_k(\alpha', s) \geq \alpha_{jk} - \alpha'_{jk} = \bar{u}_j - \lambda > u_j(\alpha, s)$. Hence, $U_j(\Pi(\alpha', s), X(\alpha', s); \alpha, s) > u_j(\alpha, s)$ and the proof of this case is complete.

Second case: $Y = Q$. By hypothesis, $\Pi(\alpha, s) \neq \bar{\rho}$. Let q_k be any seller for whom $\bar{\rho}_k > \Pi_k(\alpha, s) \geq s_k$. Given that $\bar{\rho}_k > s_k$, Proposition B implies that q_k is matched under $X(\alpha, s)$. Furthermore, for some positive λ, $\bar{\rho}_k > \bar{\rho}_k - \lambda > \Pi_k(\alpha, s)$.

Let s' be defined as follows: $s'_t = s_t$ for all $q_t \neq q_k$ and $s'_k = \bar{\rho}_k - \lambda$. That is, under the profile (α, s'), q_k replaces his true valuation by s'_k, whereas the other players keep their strategies. We now show that $V_k(\Pi(\alpha, s'), X(\alpha, s'); \alpha, s) > \Pi_k(\alpha, s) - s_k$. First note that $\bar{\rho}_t \geq s'_t$ for all q_t, which implies that $(\bar{\rho}, X(\alpha, s))$ is a feasible allocation for $M(\alpha, s')$. We use the facts that $(\bar{\rho}, X(\alpha, s))$ is a competitive equilibrium in $M(\alpha, s)$ and that if q_t is unmatched at $X(\alpha, s)$, then $\bar{\rho}_t = s_t = s'_t$, to determine that $(\bar{\rho}, X(\alpha, s))$ is a competitive equilibrium for $M(\alpha, s')$. Because $\bar{\rho}_k - s'_k = \lambda > 0$, it follows from Proposition B that q_k is matched at any optimal matching for $M(\alpha, s')$; in particular, q_k is matched under $X(\alpha, s')$. Hence, $V_k(\Pi(\alpha, s'), X(\alpha, s')) = \Pi_k(\alpha, s') - s_k \geq s'_k - s_k = (\bar{\rho}_k - \lambda - s_k) > \Pi_k(\alpha, s) - s_k$, which completes the proof. ■

Theorem 1 implies that, for any competitive equilibrium rule (Π, X) (not necessarily one of the optimal competitive equilibrium rules), if $(\Pi(\alpha, s), X(\alpha, s))$ is not the buyer (respectively, seller)-optimal competitive equilibrium for market $M(\alpha, s)$,[8] then in the induced game $\Gamma(\Pi, X)$, when the true profile of valuations is (α, s) truthful behavior is not a best response (hence, it is also not a dominant strategy) for

[8] Notice that $M(\alpha, s)$ has more than one competitive equilibrium price vector.

at least one buyer (respectively, seller), and so (α, s) is not a Nash equilibrium for $\Gamma(\Pi, X)$. Therefore, if $M(\alpha, s)$ has more than one competitive equilibrium price vector, (α, s) is not a Nash equilibrium in the induced game with true profile of valuations (α, s).

In particular, if the buyer (seller)-optimal competitive equilibrium rule is used and the true market has more than one competitive equilibrium price vector, some seller (buyer) can profitably misrepresent her (his) valuations, assuming the others tell the truth.

That is, for any competitive equilibrium rule (Π, X), if $(\Pi(\alpha, s), X(\alpha, s))$ is not the buyer (respectively, seller)-optimal competitive equilibrium for market $M(\alpha, s)$, then in the induced game $\Gamma(\Pi, X)$, truthful behavior is not a best response (hence, it is also not a dominant strategy) for at least one buyer (respectively, seller). In particular, if the seller (respectively, buyer)-optimal competitive equilibrium rule is used for a market with more than one competitive equilibrium price vector, some buyer (respectively, seller) can profitably misrepresent her (respectively, his) valuations, assuming the others tell the truth.

Hence, we have proved the following corollary:

Corollary 1. (Folk Manipulability Theorem) *Consider the buyer-optimal (respectively, seller-optimal) competitive equilibrium rule. Suppose that $M(\alpha, s)$ has more than one vector of equilibrium prices. Then there is a seller (respectively, buyer) who can manipulate the rule via $M(\alpha, s)$.*

Another immediate consequence of Theorem 1, stated in Corollary 2, is that for every market $M(\alpha, s)$ with more than one vector of competitive equilibrium prices, there is no competitive equilibrium rule such that the induced game with true profile of valuations (α, s) gives to every agent an incentive to play her/his sincere strategy. We notice that the old impossibility result (Theorem 7.3 of Roth and Sotomayor, 1990) implies that there is a particular market $M(P, Q; \alpha, s)$ such that, for all competitive equilibrium rule (mechanism) (Π, X), there are some agent y_i and some market $M(P, Q; \alpha', s')$, where (α', s') differs from (α, s) only in agent y_i's valuation, for which $U_i(\Pi(\alpha', s'), X(\alpha', s'); \alpha, s) > U_i(\Pi(\alpha, s), X(\alpha, s); \alpha, s)$ if y_i is a buyer, and $V_i(\Pi(\alpha', s'), X(\alpha', s'); \alpha, s) > V_i(\Pi(\alpha, s), X(\alpha, s); \alpha, s)$ if y_i is a seller. Corollary 2 strengthens this result by stating that this is true for all markets $M(P, Q;\ \alpha, s)$. That is, the old Impossibility Theorem implies that every competitive equilibrium rule is manipulable via *some* market (which has more than one vector of equilibrium prices), whereas our Theorem 1 implies that every competitive equilibrium rule is manipulable via *every* market that has more than one vector of equilibrium prices.

We denote $\Gamma(\Pi, X;\ \alpha, s)$ the strategic game when the competitive rule (Π, X) is applied and the true profile of valuations of the agents is (α, s), that is, $\Gamma(\Pi, X;\ \alpha, s)$ represents the game induced by the use of the competitive equilibrium rule (Π, X) for the market $M(\alpha, s)$.

Corollary 2. (Folk General Impossibility Theorem) *Suppose $M(\alpha, s)$ has more than one vector of equilibrium prices. No competitive equilibrium rule (Π, X) exists for which, in the game induced $\Gamma(\Pi, X;\ \alpha, s)$, stating (α, s) is a dominant strategy for every agent.*

We can immediately derive from the previous results that no competitive equilibrium rule for a market with more than one vector of equilibrium prices is strategy-proof; using the standard definition, a rule for $M(\alpha, s)$ is *strategy-proof* if, regardless of what the other players report, a given player always wants to report truthfully. This result, together with the Non-manipulability Theorem, implies that a competitive equilibrium rule for a buyer-seller market is strategy-proof if and only if the market has only one equilibrium price vector. Then, a rule is strategy-proof for a buyer-seller market if and only if truthful reporting is a Nash equilibrium in the game induced by the given rule. We state the results in Proposition 1.

Proposition 1. *A competitive equilibrium for a buyer-seller market rule is strategy-proof if and only if the market has only one equilibrium price vector.*

Finally, Theorem 1 allows obtaining the converse of the non-manipulability theorem of Demange and Gale (1985). We state a general result in the proposition below. We say that a rule is *strategy-proof for the buyers* (or sellers) if truthful reporting is a dominant strategy for every buyer (seller).

Proposition 2. *A competitive equilibrium rule for $M(\alpha,s)$ is strategy-proof for the buyers (respectively, sellers) if and only if the rule maps (α, s) to the buyer-optimal (respectively, seller-optimal) competitive equilibrium for $M(\alpha, s)$.*

The general manipulability theorem implies that the condition is necessary. The fact that the condition is sufficient follows from the non-manipulability theorem of Demange and Gale (1985).

5. EXTENSION AND CONCLUDING REMARKS

The model considered in the previous sections is "asymmetric" in that a seller specifies only one number, his reservation price, whereas a buyer specifies an *n*-vector, her valuations for each of the goods. There are relevant economic environments where the agents from both sides of the market care about both the identity of their partner and the monetary transfers. This situation arises if we consider sellers to be workers who are selling their services to employers. Such environments can be modeled by the *job assignment model*. In this model, each seller q_k specifies a vector of prices $\rho_k = (\rho_{1k},\ldots, \rho_{mk})$ and each buyer p_j demands the object of seller q_k at price ρ_{jk} if $\alpha_{jk} - \rho_{jk} \geq 0$ and $\alpha_{jk} - \rho_{jk} \geq \alpha_{jt} - \rho_{jt}$ for all $q_t \in Q$.

A further extension of the job assignment model is the symmetric and non-transferable utility game proposed in Demange and Gale (1985), which we will describe briefly. The preferences of the agents are given by strictly increasing utility functions that are not necessarily linear: $U_{jk}(y)$ represents the utility of p_j if she is matched with q_k and receives a monetary payment of y, and $V_{jk}(y)$ denotes the utility of q_k if he is matched with p_j and receives a monetary payment of y.[9] Moreover, for each p_j and q_k, the utility of being unmatched is given by numbers r_j and z_k, respectively.

In this general one-to-one two-sided matching model with money, we denote with ρ_k the array of prices ρ_{jk} and with ρ the *n*-tuple $(\rho_1,\ldots, \rho_n)$. If object q_k is sold to buyer p_j at price ρ_{jk}, then this buyer obtains a utility $u_j \equiv U_{jk}(-\rho_{jk})$ and the payoff of seller q_k is $v_k \equiv V_{jk}(\rho_{jt})$. The feasibility condition requires that $V_{jk}(\rho_{jk}) \geq z_k$ for all $q_k \in Q$. Buyer p_j demands the object q_k at prices ρ if $U_{jk}(-\rho_{jk}) \geq r_j$ and $U_{jk}(-\rho_{jk}) \geq$

[9] We hope not to cause any confusion in using the same notation for the utility function and the true payoff. The domains of these functions are different.

$U_{jt}(-\rho_{jt})$ for all $q_t \in Q$.

The competitive equilibrium concept in this model is defined as usual. It is a matter of verification that if $\rho = (\rho_1,\ldots, \rho_n)$ is a profile of feasible price vectors and x is a feasible matching, then (ρ, x) is a competitive equilibrium if and only if the corresponding outcome $(u, v; x)$ is in the core.[10] The seller-optimal equilibrium prices $\overline{\rho} = (\overline{\rho}_1,\ldots, \overline{\rho}_n)$ correspond to the seller-optimal core payoff $(\underline{u}, \overline{v})$, and the buyer-optimal equilibrium prices $\underline{\rho} = (\underline{\rho}_1,\ldots, \underline{\rho}_n)$ are symmetrically defined. Finally, a competitive equilibrium rule is a function that produces, for any market that keeps fixed P and Q, a competitive equilibrium allocation for the given market.

How would this greater generality affect our results? Actually, nothing needs to be changed in the statement of Theorem 1 (after the necessary adaptations), which holds for the general two-sided matching model with money as a continuous variable. In the general model, an agent can change her/his utility or reservation payoff or both. Thus, the proof of this extension of the general manipulability theorem follows the lines of the second part of the proof presented in the previous section. The formal proof is given in the Appendix.

The Non-manipulability Theorem suggests that the buyers will always play their sincere strategy if the buyer-optimal competitive equilibrium rule is used. The Folk Manipulability Theorem (and the General Manipulability Theorem) implies that playing sincerely is often not the optimal strategy for at least one seller. It is thus natural to look for a profile of strategies with the property that, once it is selected, no seller will have an incentive to change his strategy while the other agents do not change theirs. This problem was first considered in Demange and Gale (1985). These authors analyze the strategic equilibrium of the game induced by the rule that produces the buyer-optimal competitive equilibrium payoff if the buyers always play their sincere strategies and if the sellers keep fixed their utility function and only manipulate their reservation prices. The analysis of the case in which the rule produces any competitive equilibrium payoff and there is no restriction on the strategies selected by the agents is addressed in Sotomayor (2011). Related results are proved for the marriage market in Roth and Sotomayor (1990) and for the College Admission market in Sotomayor (2012).

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[10] The core of this market is characterized as the set of outcomes $(u, v; x)$ such that, for all $(p_j, q_k) \in P\text{x}Q$, we have that (a) $u_j \geq r_j$ and $v_k \geq z_k$; (b) $U_{jk}^{-1}(u_j) + V_{jk}^{-1}(v_k) \geq 0$; (c) $U_{jk}^{-1}(u_j) + V_{jk}^{-1}(v_k) = 0$ if $x_{jk} = 1$ and (d) $u_j = r_j$ if p_j is unmatched at x, and $v_k = z_k$ if q_k is unmatched at x.

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APPENDIX

Proof of the General Manipulability Theorem for the general two-sided matching model with money as a continuous variable

We will use the following propositions, which are implied by Proposition 9.11 and Theorem 9.8, presented in Roth and Sotomayor (1990), due to Demange and Gale (1985).

Proposition C. *Let* $(\underline{u}(r, z), \overline{v}(r, z))$ *and* $(\underline{u}(r, z^o), \overline{v}(r, z^o))$ *be the seller-optimal competitive equilibrium payoffs for* $M = (P, Q, U, r; V, z)$ *and* $M' = (P, Q, U, r; V, z^o)$*, respectively. If* $z^o \geq z$*, then,* $\overline{v}_k(r, z^o) \geq \overline{v}_k(r, z)$ *for all* $q_k \in Q$*.*

Proposition D. *Let* $(u, v; x)$ *and* $(u^o, v^o; x^o)$ *be competitive equilibrium outcomes. If* $v_k > z_k$*, then* q_k *is matched under* x *and* x^o*.*

For the proof of the theorem, we consider that P, Q, U and V are fixed, then the markets $M = (P, Q, U, r'; V, z')$ can be denoted by $M(r', z')$. Let $M \equiv M(r, z)$ be the true market. Let $(u, v; x)$ be the competitive equilibrium outcome corresponding to the competitive equilibrium (ρ, x) produced when the rule is applied to the true market M. Let $(\underline{u}(r, z), \overline{v}(r, z))$ be the seller-optimal competitive equilibrium payoff for M, and let $\overline{\rho}(r, z)$ be the seller-optimal equilibrium prices. Let x^* be a compatible optimal matching.

We prove the result for $Y = Q$; the proof for $Y = P$ follows dually. By hypothesis, $\rho \neq \overline{\rho}(r, z)$; thus, $v \neq \overline{v}(r, z)$ because the functions V_{jk} are strictly increasing. Let q_k be any seller such that $\overline{v}(r, z)_k > v_k \geq z_k$. Then, q_k is matched under x^* by Proposition D. Moreover, for some positive λ, $\overline{v}_k(r, z) > \overline{v}_k(r, z) - \lambda > v_k \geq z_k$.

We show that seller q_k can profitably misrepresent his reservation payoff, assuming the other agents tell the truth. Let z' be a profile of reservation payoffs for the sellers where $z'_t = z_t$ for all $q_t \neq q_k$ and $z'_k = \overline{v}_k(r, z) - \lambda$. That is, in the market $M(r, z')$, q_k replaces his true reservation payoff z_k by z'_k and the other agents keep their true reservation payoffs. We show that the true payoff of q_k under $(\Pi(r, z'), X(r, z'))$ – the allocation produced by the rule (Π, X) – is greater than v_k. To observe this fact, first note that because $z' \geq z$, Proposition C implies that $\bar{v}_k(r, z') \geq \bar{v}_k(r, z) > z'_k$, where $(\underline{u}(r, z'), \bar{v}(r, z'))$ is the seller-optimal competitive equilibrium payoff for $M(r, z')$. Then, $\bar{v}_k(r, z') > z'_k$ and, by Proposition D**,** q_k is matched at any matching compatible with a competitive equilibrium payoff for $M(r, z')$; in particular, q_k is matched under $X(r, z')$. Therefore, the true payoff of q_k under $(\Pi(r, z'), X(r, z'))$ is $v_k(r, z')$, which, together with $v_k(r, z') \geq z'_k = \bar{v}_k(r, z) - \lambda > v_k$, completes the proof. ■