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Abstract:

In the one-sided Assignment game any two agents can form a partnership. If this is done, the partners undertake some joint activity, which produces a gain that is split between them. We approach this model by focusing on simple outcomes - feasible and individually rational outcomes where only unmatched agents can block. We prove that this blocking can be done in such a way that the payoffs from the trades done are not changed as players reach the core. The core is non-empty iff every simple and unstable outcome can be extended to a simple outcome by a sequence of adjustments in which, at each step, payoffs are preserved for agents already matched and increased only for those newly matching. Hence, starting from the simple outcome where everybody stands alone, we can gradually increase cooperation by making Pareto improvements (and still staying within simple outcomes), until we reach the core, or until the payoff cannot be simple anymore. That is, increase in payoffs is only available through non-optimal cooperation of some agents. In addition, the total sum of these payoffs is the same at any core outcome. The gains in insight with this approach allows a necessary and sufficient condition for the non-emptiness of the core to be identified. Several properties of the core outcomes of economic interest are proved.

Keywords: matching, assignment game, core, Pareto optimal simple outcome

JEL Codes: C78, D78

THE ROLE PLAYED BY SIMPLE OUTCOMES IN COALITION FORMATION PROCESS OF THE CORE OUTCOMES

by

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ABSTRACT

In the one-sided Assignment game any two agents can form a partnership. If this is done, the partners undertake some joint activity, which produces a gain that is split between them. We approach this model by focusing on simple outcomes - feasible and individually rational outcomes where only unmatched agents can block. We prove that this blocking can be done in such a way that the payoffs from the trades done are not changed as players reach the core. The core is non-empty iff every simple and unstable outcome can be extended to a simple outcome by a sequence of adjustments in which, at each step, payoffs are preserved for agents already matched and increased only for those newly matching. Hence, starting from the simple outcome where everybody stands alone, we can gradually increase cooperation by making Pareto improvements (and still staying within simple outcomes), until we reach the core, or until the payoff cannot be simple anymore. That is, increase in payoffs is only available through non-optimal cooperation of some agents. In addition, the total sum of these payoffs is the same at any core outcome. The gains in insight with this approach allows a necessary and sufficient condition for the non-emptiness of the core to be identified. Several properties of the core outcomes of economic interest are proved.

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INTRODUCTION

In a decentralized setting in which players are permitted to freely communicate, can interact with each other and get together in groups by making binding agreements, and preferences over outcomes, as well as the rules of the game are common knowledge, the game theoretic predictions are that only core allocations will occur.

Under this context, the history underlying a core outcome makes no difference at all. It is a matter for the players to work out for themselves in order to get a specific agreement.

This paper aims to give some new conceptual view of the cooperative games and coalition formation process in a core outcome. For the sake of simplicity we consider a simple one-sided assignment model, which will be introduced in the text. This model can be viewed as an extension of the two-sided Assignment game of Shapley and Shubik (1972) to the case where any two agents can interact by forming a partnership and then splitting, between them, the surplus generated by the pair, in any way they agree. However, the analysis also holds if, instead of partnerships, the agents form coalitions of any size.

To start, imagine the following scenery in which the coalitional interactions take place sequentially, along several stages. Non-trading agents at some stage may become trading agents at a subsequent stage.

The basic assumption is that agents only engage in cooperation which is optimal for them, in the sense that two agents will only emerge with an agreement if both believe that more favorable terms will not be obtained in any future negotiations. Consequently, if at some step some trading agent breaks his/her present contract to make a new one with some other agent, both agents will be indifferent between the current contract and the terms of the new contract. That is, once a transaction is done at a given stage, the agents involved will keep their payoffs (even if they dissolve their current partnerships to enter new ones) at the subsequent stages. Therefore, after each coalitional interaction, the resulting outcome has the property that increase in payoffs is only available to those agents who are not trading at this stage but will be able to trade among them at a subsequent stage. This means that, at any stage of the process, no pair of trading agents can block the current outcome.

The current outcomes in each step of this process are *simple outcomes*. These are feasible and individually rational outcomes in which none of the matched players is member of a blocking pair[2]. The set of simple outcomes is non-empty, since the outcome where every one is unassigned is simple. Clearly, the core outcomes are simple.

A simple and intuitive understanding of the dynamics of the coalition formation in the core outcomes is then propitiated through the following adjusting process of simple outcomes. Starting from the simple outcome where everybody stands alone, we can gradually increase cooperation by making Pareto improvements and still staying within simple outcomes, until no transaction is able to benefit the agents involved, in whose case we reach the core, or until the outcome cannot be simple anymore. That is, increase in payoffs is only available through non-optimal cooperation of some agents.

Our main result confirms this intuition and fundaments the conclusions above. We prove that, indeed, if any simple outcome that is not in the core can be extended to a simple outcome, by keeping the payoffs of the trading agents and increasing the payoff of at least one agent, then the core is non-empty. This follows trivially from the fact that a simple outcome which is Pareto optimal among all simple outcomes always exist. What is not proved so simply is that this condition is also necessary for the non-emptiness of the core. That is, under the assumption of the non-emptiness of the core, every simple and unstable outcome can be extended to a simple outcome.

Therefore, whenever outcomes of the core exist, in each step of the process, the first group of trading agents constrains the final payoffs of its members, and each subsequent group to form does the same for its members. Somehow, something prevents the constraints arising from the first group of trading agents from being inconsistent with a core outcome. However, the core depends on information about all players' preferences. The question is then if the dynamic process is truly decentralized, in the sense that it allows players to act based on information they can reasonably be expected to have available, information about their own and their prospective partner's preferences. Or if there is a stronger, hidden assumption here.

If T is a group of trading agents formed at some step of a given coalition formation process, then it need not be a group of trading agents formed at some step of a different coalition formation process. An interesting feature of the simple outcomes is that, in spite

[2] This is the continuous version for the concept introduced in Sotomayor (2008) for the discrete one-sided matching model.

of this, the payoffs of these agents will have the same sum in every coalition formation process.

It is intuitive that, if the core is empty, then there is some stage at which any new trade requires that some of the agents involved is not behaving optimally. On the other hand, if new trades always occur at each stage along the process, until that no transaction be able to benefit the agents involved, the final outcome is a core outcome. In fact, if this condition is satisfied, no simple and unstable outcome is Pareto optimal among all simple outcomes. The result then follows from the proof that such Pareto optimal outcomes always exist.

In practical terms, simple outcomes provide an economic intuition of how blocking can be done by non-trading agents when the core is non-empty. Of course, a simple outcome out of the core is blocked by at least one pair of non-trading agents. If the core is non-empty, this blocking can be done in such a way that the payoffs from the trades done need not be changed if players reach the core. In addition, as it is proved here, the total sum of these payoffs is the same at any stable outcome.

For the one-sided assignment game, our approach also allows us to capitalize on the properties of simple outcomes to draw conclusions about the core outcomes. These are well known properties of the core for the two-sided assignment game. The fact that these properties persist for the core of the one-sided market suggests that they may even be more fundamental than previous results have suggested.

The notion of simple outcome given here provides us with a new conceptual view of the cooperative games and coalition formation process. We hope that this concept will be helpful in understanding both dynamics of the coalition formation and the structure of core outcomes.

This paper is organized as follows. In section 2 we describe the model and present the preliminary results and definitions. Section 3 introduces the concept of simple outcome, proves some of its properties and gives the conceptual framework to be used in section 4. Section 4 is devoted to the existence and non-existence of stable outcomes. Section 5 concludes the paper and discusses related work.

2. THE FORMAL ONE-SIDED ASSIGNMENT MODEL AND SOME PRELIMINARIES

The description of this model follows the one given in Roth and Sotomayor (1990) for the case with two sides, with the appropriate adaptations. There is a finite set of players, $N=\{1,2,...,n\}$. Associated to each partnership $\{i,j\}$ there is a nonnegative real number $a_{\{i,j\}}$ which will be denoted a_{ij}. A game in coalitional function form with side payments is determined by (N,a), with the numbers a_{ij} being equal to the worth of the coalitions $\{i,j\}$. The worth of large coalitions is determined entirely by the worth of the pairwise combinations that the coalition members can form. That is, the coalitional function v is given by

$v(S)=0$ *if* $|S|=1$;

$v(S)=a_{ij}$ if $S=\{i,j\}$;

$v(S)=max\{v(i_1,j_1)+ v(i_2,j_2)+...+ v(i_k,j_k)\}$[3] for arbitrary coalitions S, where k is an integer number that does not exceed the integer part of $|S|/2$. The maximum is taken over all sets $\{i_1,j_1\},...,\{i_k,j_k\}$ of k distinct pairs in S.

Thus, the rules of the game are that any pair of agents $\{i,j\}$ can together obtain a_{ij}, and any larger coalition is valuable only insofar as it can organize itself into such pairs. The members of any coalition may divide among themselves their collective worth in any way they like. We might think of the two-sided Assignment game of Shapley and Shubik (1972) as being a particular case of this game, by taking $N=P\cup Q$, $P\cap Q=\phi$, $v(S)=0$ if S contains only agents of P or only agents of Q.

We will consider that any player i can be self-matched and will define $a_{ii}=0$ for all $i\in N$. Then,

Definition 1. *A **feasible matching** x is a one-to-one correspondence from N onto itself of order two (that is, $x^2(j)=j$). We refer to $x(j)$ as the partner of j at x.* If $x(i)=i$ we say that i is **unmatched** at x.

For our purposes it is simpler to define,

[3] We write $v(i,j)$ rather than $v(\{i,j\})$.

Definition 2. *The vector* u, *with* $u \in R^n$, *is called a* ***feasible payoff*** *for* (N,a) *if there is a feasible matching* x *such that*

$$u_i + u_j = a_{ij} \text{ if } x(i)=j \text{ and } u_i=0 \text{ if } x(i)=i.$$

In this case we say that (u,x) *is a feasible outcome and* x *is compatible with* u..

Remark 1. Given a coalition S, the definition of v implies that there is some feasible matching x such that $x(S)=S$ and $\sum_{i\in S,\ i\leq x(i)} a_{i,x(i)}=v(S)$. Furthermore, $v(S) \geq \sum_{i\in S,\ i\leq x'(i)} a_{i,x'(i)}$ for all feasible matching x' such that $x'(S)=S$. Then, it follows from Definition 2 that $\sum_{i\in S} u_i \leq v(S)$ **for all** $S\subseteq N$ **and feasible outcome** (u,x) **with** $x(S)=S$. **In particular,** $\sum_{i\in N} u_i \leq v(N)$.

The key definition is that of stability. The general definition of stability is given in Sotomayor (2009). For the model we are treating here it is equivalent to the following:

Definition 3. *The feasible payoff* u *is* ***stable*** *if*

(i) $u_i\geq 0$, *for all* $i\in N$,

(ii) $u_i + u_j \geq a_{ij}$ *for all* $\{i,j\}\subseteq N$.

If x *is compatible with* u *we say that* (u,x) *is a stable outcome.*

Condition (i) (individual rationality) means that a player always has the option of remaining unmatched. Condition (ii) is the natural one: If it is not satisfied for some agents i and j, then it would pay them to break up their present partnership(s) and form a new one together, because this could give them each a higher payoff. In this case, we say that $\{i,j\}$ blocks u.

The following example shows that the set of stable outcomes may be empty.

Example 1. Consider $N=\{1,2,3\}$ and $a_{ij}=1$ for all $\{i,j\}\subseteq N$. For every feasible payoff u there will exist two players i and j such that $u_i+u_j<1$. Hence, the set of stable outcomes of this game is empty.

Definition 4. *The payoff* u *is in the* **core** *of* (N,a) *if* $\sum_{i\in N} u_i=v(N)$ *and* $\sum_{i\in S} u_i \geq v(S)$ *for all* $S\subseteq N$.

Proposition 1. *The set of stable payoffs equals the core of (N,a).*

Proof. Suppose u is a stable payoff. Then, u is feasible and so

$$\sum_{i\in N} u_i \leq v(N), \qquad (1)$$

by Remark 1. Given a coalition S, let y be a feasible matching such that $y(S)=S$ and $v(S)=\sum_{i\in S,\ i\leq y(i)} a_{\{i,y(i)\}}$. The stability of u implies that $u_i + u_{y(i)} \geq a_{i,y(i)}$ for all $i\in S$, so

$$\sum_{i\in S} u_i \geq v(S) \text{ for all coalition } S. \qquad (2)$$

By (1) and (2) it follows that $\sum_{i\in N} u_i=v(N)$ and $\sum_{i\in S} u_i \geq v(S)$ for all $S\subseteq N$, so u is in the core.

Now, suppose u is in the core. Definition 4 implies that $u_i + u_j \geq v(i,j)=a_{ij}$ for every coalition $\{i,j\}$ and $u_i \geq v(i)=0$ for all $i\in N$, so u does not have any blocking pair and is individually rational. To see that u is feasible let x be a feasible matching such that $v(N)=\sum_{i\leq x(i)} a_{\{i,x(i)\}}$. Now use that $\sum_{i\in N} u_i=v(N)$ and $u_i + u_{x(i)} \geq a_{i,x(i)}$ for all $i\in N$, to get that $\sum_{i\in N} u_i=\sum_{i\leq x(i)}a_{\{i,x(i)\}}.\leq\sum_{i\leq x(i)} (u_i+u_{x(i)})=\sum_{i\in N} u_i$, so the inequality cannot be strict and so $u_i+u_{x(i)} = a_{i,x(i)}$ for all $i\in N$. Since $u_i\geq 0$, it follows that $u_i=0$ if $x(i)=i$. Hence u is stable and the proof is complete.

Definition 5. *The feasible matching* x *is optimal if* $\sum_{i\leq x(i)} a_{\{i,x(i)\}} = v(N)$.

The following two propositions make clear why, similarly to the two-sided Assignment game and in contrast to the discrete version (roommate-problem), we can concentrate on the payoffs to the agents rather than on the underlying matching.

Proposition 2. *If* x *is an optimal matching, then it is compatible with any stable payoff* u.

Proof. Immediate from the fact that if u is a stable payoff then $u_i + u_{x(i)} \geq a_{i,x(i)}$ for all $i\in N$, so $v(N)=\sum_{i\in N} u_i=\sum_{i\leq x(i)} (u_i + u_{x(i)})\geq \sum_{i\leq x(i)} a_{\{i,x(i)\}}=v(N)$, where the last equality is implied by the optimality of x. But then, the inequality cannot be strict and $u_i + u_{x(i)}$

$=a_{\{i,x(i)\}}$ for all $i \in N$. Since $u_i \geq 0$, it follows that $u_i=0$ if $x(i)=i$. Hence, x is compatible with u.

Proposition 3. *If (u,x) is a stable outcome, then x is an optimal matching.*

Proof. Immediate from the fact that

$v(N)=\sum_{i\in N} u_i=\sum_{i\leq x(i)} (u_i + u_{x(i)})= \sum_{i\leq x(i)} a_{\{i,x(i)\}}$.

3. SIMPLE OUTCOMES: CONCEPTUAL FRAMEORK AND PROPERTIES

The feasible and individually rational outcome (u,x) is **simple** if every player is unmatched at x or, in case $x(i)\neq i$ for some i, then i is not part of any blocking pair of (u,x). Hence, in case a blocking pair $\{i,j\}$ exists, i and j are unmatched at x. Since the outcome where everyone is unmatched is simple, **the set of simple outcomes is non-empty.** Clearly, every stable outcome is simple. If (u,x) is a simple outcome we say that u is a simple payoff.

The key lemma is:

Lemma 1. *Let (u,x) be a simple outcome and let (w,y) be a stable outcome. Let $T=\{j \in N;\ x(j)\neq j\}$, $M_u=\{j \in N;\ u_j>w_j\}$, $M_w=\{j \in T;\ w_j>u_j\}$ and $M_0=\{j \in T;\ u_j=w_j\}$. Then $x(M_u)= y(M_u)=M_w$ and $x(M_w)=y(M_w)=M_u$. Furthermore, $x(M_0)=M_0$.*

Proof. All j in M_u are matched under x, since $u_j>w_j\geq 0$. Analogously, all j in M_w are matched under y, since $w_j>u_j \geq 0$. If j is in M_u then $k=x(j)$ is in M_w, for if not

$$a_{jk} = u_j + u_k > w_j + w_k$$

which implies that (j,k) blocks (w,y), contradiction. Simmilarly, if k is in M_w then $j=y(k)$ is in M_u, for if not (j,k) blocks x, but k is matched under x, which contradicts the fact that (u,x) is simple. Therefore, $x(M_u)\subseteq M_w$ and $y(M_w)\subseteq M_u$, so $M_u\subseteq x(M_w)$ and $M_w\subseteq y(M_u)$. It follows that

$$|M_u|=|x(M_u)|\leq|M_w|=|y(M_w)|\leq|M_u| \text{ and } |M_w|\leq|y(M_u)|=|M_u|\leq|x(M_w)|=|M_w|,$$

which implies $x(M_u)=M_w$, $y(M_w)=M_u$, $y(M_u)=M_w$ and $x(M_w)=M_u$. The last part of the lemma follows from the fact that $x(M_0)\subseteq M_0$ and x is one-to-one. Hence the proof is complete.

An immediate consequence of this lemma is

Proposition 4. *Let (u,x) and (w,y) be stable outcomes. If j is matched to k under x or under y and $u_j>w_j$ then $w_k>u_k$.*

If j has a positive payoff under a simple outcome (u,x) then he/she is matched under every stable outcome. In fact,

Proposition 5. *Let (u,x) be a simple outcome and let (w,y) be a stable outcome. If j is unmatched under y, $u_j=0$.*

Proof. Suppose j is unmatched under y. If $u_j>0=w_j$ define M_u as in Lemma 1. Then $j\in M_u$, so Lemma 1 implies j is matched under y, which is a contradiction.

The fact that every stable outcome is simple implies that **if *j* is unmatched under a stable outcome then he/she gets payoff zero under any stable payoff.**[4]

In what follows we will make use of the following definition:

Definition 6. *Let (u,x) be a simple outcome. Let $T=\{j\in N;\ x(j)\neq j\}$. We say that the feasible outcome **(u^*,z) extends (u,x)** if $u^*_j>u_j$ for some $j\notin T$ and $u_j=u^*_j$ for all $j\in T$. If (u^*,z) is simple (respectively stable) then (u^*,z) is said to be a simple (respectively stable) extension of (u,x).*

Given a simple outcome (u,x), which is not in the core, we can always obtain a new outcome (u^*,z) that keeps the payoffs of the matched players. What is new is that the outcome (u^*,z) can be constructed so that it is stable.

Proposition 6. *Let (u,x) be an unstable and simple outcome. Suppose the set of stable outcomes is non-empty. Then there exists a stable outcome (u^*,z) that extends (u,x).*

[4] This result was proved in Demange and Gale (1985) for a two-sided matching market where the utilities are continuous, so it applies to the two-sided Assignment game of Shapley and Shubik (1972).

Proof. Let (w,y) be a stable outcome. Define M_u, M_w and M_0 as in Lemma 1. Let $T=\{j\in N;\ x(j)\neq j\}$. Then $T=M_u\cup M_w\cup M_0$. Now construct the outcome (u^*,z) as follows: $z(j)=x(j)$ and $u^*_j=u_j$ if $j\in M_u\cup M_w$; $z(j)=y(j)$ and $u^*_j=w_j$ otherwise. It follows from Lemma 1 that all of $M_u\cup M_w$ are matched among themselves under y, so z is feasible. We are going to show that

$u^*_j\geq u_j$ for all $j\in N$; (1)

and $u^*_j=u_j$ for all $j\in T$. (2)

In fact, if $j\notin M_u\cup M_w$ then $j\in M_0$ or $x(j)=j$. In the first case, $u_j=w_j=u^*_j$ and in the other case $u_j=0\leq w_j=u^*_j$. Therefore, $u^*_j=u_j$ for all $j\in T$ and $u^*_j\geq u_j$ for all $j\in N$. We claim that (u^*,z) is stable. That (u^*,z) is feasible and individually rational is immediate from the feasibility and individual rationality of (u,x) and (w,y). Thus, it remains to show that (u^*,z) does not have any blocking pair. The fact that (u,x) is simple and (w,y) is stable implies that $\{j,k\}$ does not block (u^*,z) in the following cases: $\{j,k\}\subseteq M_u\cup M_w$ and $\{j,k\}\subseteq N-[M_u\cup M_w]$. Then, without loss of generality, suppose $j\in M_u\cup M_w$ and $k\in N-[M_u\cup M_w]$. If $\{j,k\}$ blocks (u^*,z), we must have $a_{jk}> u^*_j+u^*_k \geq u_j+u_k$, where in the last inequality we used (1). In this case (j,k) would block (u,x). However, $j\in M_u\cup M_w$, so j is matched at x, which contradicts the fact that (u,x) is simple. Hence, in any case, (u^*,z) does not have any blocking pair, so it is stable.

To see that (u^*,z) extends (u,x), use (1) and (2) to conclude that $u^*_i\geq u_i$ for all $i\in N$. The fact that (u^*,z) is stable, (u,x) is unstable and $u^*_j=u_j$ for all $j\in T$ implies that there is some $j\notin T$ such that $u^*_j>u_j$. Hence the proof is complete.

The following proposition asserts that the sum of the payoffs of the matched agents is the same under every stable outcome.

Proposition 7. *Let (u,x) be a simple outcome and let (w,y) be in the core. Let $T=\{j\in N;\ x(j)\neq j\}$. Then, $\sum_{i\in T} u_i =\sum_{i\in T} w_i$.*

Proof. It follows from Remark 1 and from the fact that T does not block (u,x) that $\sum_{i\in T} u_i=v(T)$. Define the stable outcome (u^*,z) as in the proof of Proposition 7. Then, $v(N)=\sum_{i\in N} u^*_i = \sum_{i\in T} u_i +\sum_{i\in N-T} w_i = v(T) +\sum_{i\in N-T} w_i \leq \sum_{i\in T} w_i + \sum_{i\in N-T} w_i =\sum_{i\in N} w_i =$

$v(N)$, so $v(T)=\sum_{i\in T} w_i$, where in the inequality was used that T does not block (w,y).. Hence, $\sum_{i\in T} u_i = \sum_{i\in T} w_i$, and the proof is complete.

4. EXISTENCE OF STABLE MATCHINGS

Theorem 10 asserts that the condition that every simple and unstable outcome has a simple extension is necessary and sufficient for the non-emptiness of the core. We need one more concept.

Definition 7. *The payoff u is a* ***Pareto optimal simple payoff (PS for short)*** *if it is simple and there is no simple payoff w such that:*

(i) $w_j \geq u_j$ *for all players j and*

(ii) $w_j > u_j$ *for at least one player j.*

If x is compatible with u, we say that (u,x) is a Pareto optimal simple outcome.

Therefore, if u is PS and $w_j > u_j$ for some player j and some simple payoff w, there is some other player k such that $w_k < u_k$. Proposition 9 asserts that the set of simple payoffs is a compact set of R^n. Then, there is some simple payoff u^* such that $\sum_{j\in N} u^*_j \geq \sum_{j\in N} u_j$ for all simple payoffs u. Clearly, u^* is a Pareto optimal simple payoff.

Proposition 8. *The set of simple payoffs is a compact set of R^n.*

Proof. The set of simple payoffs is bounded, since $0 \leq u_j \leq v(N)$, for all $j \in N$ and all simple payoff u. To see that it is closed, take any sequence $(u^t)_t$ of simple payoffs, with $u^t \to u$, when t tends to infinity. Since the set of matchings is finite, there is some matching x , which is compatible with infinitely many terms of the sequence u^t. We will use the same notation (u^t) for this subsequence. Then, if $x(j)=k$, $u_j + u_k = lim_{t\to\infty}(u^t_j + u^t_k) = lim_{t\to\infty} a_{jk=}a_{jk}$; if $x(j)=j$ then $u_j = lim_{t\to\infty} u^t_j = 0$. Thus, x is compatible with u, so (u,x) is feasible. Now, observe that if j is matched at x then j is not part of a blocking pair of (u,x). In fact, $u_j + u_k = lim_{t\to\infty}(u^t_j + u^t_k) \geq lim_{t\to\infty} a_{jk=}a_{jk}$. Therefore, (u,x) is simple, so u is a simple payoff. Hence, the set of simple payoffs is bounded and closed, so it is compact.

We can now prove our main result.

Theorem 1. *The set of stable outcomes is non-empty if and only if every unstable and simple outcome has a simple extension.*

Proof. If the set of stable outcomes is non-empty the result follows immediately from Proposition 7. In the other direction, let (u,x) be a Pareto optimal simple outcome. We are going to show that (u,x) is stable. In fact, suppose by way of contradiction that (u,x) is unstable. By hypothesis, there is some simple outcome (u^*,z), which extends (u,x). Then, $u^*_j \geq u_j$ for all $j \in N$ and $u^*_j > u_j$ for at least one player j. But this contradicts the fact that (u,x) is a PS outcome. Hence (u,x) is stable and the proof is complete.

5. CONCLUDING REMARKS AND RELATED WORK

This paper deals with a new generalization of the two-sided Assignment game of Shapley and Shubik (1972)[5] to the case where any two agents can form a partnership. It provides new results and a new point of view through the concepts of simple outcome and Pareto optimal simple outcome. Simple outcomes capture a sort of dynamic flavor to coalition formation, without an explicit model of dynamics. Stable outcomes are simple and Pareto optimal, but not all Pareto optimal outcomes are stable or Pareto optimal simple. Unstable outcomes may be Pareto optimal and unstable simple outcomes may be Pareto optimal simple. However, we proved that the core is non-empty if and only if no unstable simple outcome is Pareto optimal simple. Our main finding is that this result can be proved in quite elementary notions and arguments.

We idealized a decentralized procedure operating sequentially, where, at each stage, if it is possible for some group of agents to "interact", that is, to agree about their payoffs under the premise of optimal behavior, then a simple outcome results. The procedure ends when no interaction is able to benefit the agents involved or when any new interaction requires that some of the agents involved do not behave optimally. In the first case a core outcome is reached and in the other case the core is empty. Furthermore, when

[5] You can also see Roth and Sotomayor (1990) where an overview of the two-sided Assignment game is presented.

the core is non-empty, each particular core outcome is associated to a sequence of simple outcomes produced at each stage of the process. The particular core outcome is reached at the end of the process and naturally extends any simple outcome of the sequence. If the game ends without reaching the core then the core is empty.

Therefore, whenever outcomes of the core exist, in each step of the process, the first group of trading agents constrains the final payoffs of its members, and each subsequent group to form does the same for its members. Somehow, something prevents the constraints arising from the first group of trading agents from being inconsistent with a core outcome. However, the core depends on information about all players' preferences. The main feature of this process is that it is truly decentralized, in the sense that it allows players to act based on information they can reasonably be expected to have available, information about their own and their prospective partner's preferences.

The theory of simple outcomes has been used in several other models. Its main feature has been to allow that results be proved without the framework of sophisticated mathematical tools. A natural adaptation of the concept of simple outcome was introduced in Sotomayor (2008) and in Sotomayor (2005). The former paper proves that the stability condition is also necessary and sufficient for the non-emptiness of the core of the Roommate model of Gale and Shapley (1962), the non-transferable utility version of our game. In the latter, this condition is proved to be always satisfied for the Housing market with strict preferences of Shapley and Scarf (1974); so the core is always non-empty in this model.

As for two-sided markets, a version of the concept of simple outcome has been introduced in: a) Sotomayor (1996), for the Marriage market of Gale and Shapley (1962); b) Sotomayor (1999), for the College Admission model of Gale and Shapley (1962) and the discrete many-to-many matching model with substitutable and non-strict preferences; and in c) Sotomayor (2000), for the two-sided Assignment game of Shapley and Shubik (1972) and the unified two-sided matching model of Eriksson and Karlander (2000). In each of these models a non-constructive existence proof of pairwise-stable outcomes has been provided by showing that simple outcomes that are pairwise unstable are not Pareto optimal simple outcomes.

The present study aims to investigate the role played by the simple outcomes in the existence problem of core outcomes. Simple outcomes capture a sort of dynamic flavor to coalition formation, without an explicit model of dynamics. Our main finding is that a necessary and sufficient condition for the existence of core allocations can be given in quite elementary notions and arguments. We postulate a decentralized procedure operating sequentially, where, at each stage, if it is possible for some group of agents to "interact", that is, to agree about their payoffs under the premise of optimal behavior, then a simple outcome results. The procedure ends when no interaction is able to benefit the agents involved or when any new interaction requires that some of the agents involved do not behave optimally. In the first case a core outcome is reached and in the other case the core is empty. Furthermore, when the core is non-empty, each particular core outcome is associated to a sequence of simple outcomes produced at each stage of the process. The particular core outcome is reached at the end of the process and naturally extends any simple outcome of the sequence. If the game ends without reaching the core then the core is empty.

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Bondareva (1963) and Shapley (1967) proved that the core of a transferable utility game is non-empty if and only if the game is balanced. Thus, for the game considered here, our condition is equivalent to that of balancedness. This suggests to ask if this equivalence persists for all TU games. The answer to this question is not so simple. Recall that our results follow from Lemma 4, which strongly uses that there is a feasible matching underlying every feasible outcome. However, players do not necessarily form partnerships in the general game, so Lemma 4 does not always apply. On the other hand, the intuition behind a simple outcome is not related to a matching and seems to be quite general: if all "interactions" are made under the premise of optimal behavior, a simple outcome results. This suggests that, by conveniently adapting the concept of simple outcome, Lemma 4 can be avoided and so the desired equivalence can be obtained for a general TU game.

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