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by

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ABSTRACT

We prove two Folk Theorems which, together with the Non-Manipulability Theorem (Demange (1982) and Leonard (1983)), have stimulated the development of the theory on incentives for the one-to-one two-sided matching models with money as a continuous variable.

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INTRODUCTION

In the one-to-one two-sided matching models with money as a continuous

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variable, the structure of agents' preferences is given by utility functions that are increasing and continuous from $\boldsymbol{R}$ onto[2] $\boldsymbol{R}$. The special case, in which the utility functions are linear, was introduced in Shapley and Shubik (1972). The general model is due to Demange and Gale (1985). For simplicity of exposition we will concentrate on the linear case. Nevertheless, as we remark in section 5, all the results presented here hold for the general continuous model.

In these models the agents can be thought of as being buyers and sellers. In the linear case each seller owns one indivisible good and no buyer is interested in acquiring more than one item. Each buyer is assumed to place a monetary value on each of the objects. Each seller places a monetary value on his own object, that can be thought of as his reservation price. We will first assume that every seller specifies the same reservation price to every buyer. This assumption will be relaxed later. This model will be referred as *buyer-seller market*.

As in a exchange economy, given a vector of prices, every buyer will demand the set of objects that maximize his surplus, the difference between his valuation and the price of the item, assuming that this surplus is non-negative. Sellers will want to sell if the given prices exceed their valuation. A competitive equilibrium is a vector of prices, one price for each object, and an allocation of the objects to the buyers, such that the demand of every buyer is satisfied, the price of every unsold object is its reservation price and no two buyers get the same object. The vector of prices is called equilibrium price.[3] A competitive equilibrium allocation is a competitive equilibrium plus a feasibly defined payoff for each buyer.

The results proved and stated here are established for the competitive approach, but all of them can be translated, with no loss, to the cooperative approach. This is because in the cooperative approach of this model the cooperative equilibrium is given by the core, which coincides with the set of competitive equilibrium allocations (Shapley and Shubik, 1972).

The set of equilibrium prices is non-empty (Gale, 1960) and it is a complete lattice whose extreme points are the minimum equilibrium price and the maximum equilibrium price (Shapley and Shubik, 1972). These equilibrium prices are called

[2] The assumption that the utility functions are onto R implies that a sufficient monetary payment can make any match more desirable than any other match at a given payment.

[3] The price equilibrium model was presented by the first time in Gale (1960), where it was defined and proved the existence of "equilibrium prices" . In Demange, Gale and Sotomayor (1986) it was introduced the term "minimum competitive price". In Roth and Sotomayor (1990) the minimum competitive prices are called "minimum equilibrium prices".

buyer-optimal and seller-optimal equilibrium prices, respectively.

This note concentrates on a class of mechanisms whose domain is the set of buyer-seller markets. For each market of this domain, such a mechanism selects a competitive equilibrium. This competitive equilibrium is not necessarily the optimal for the buyers or the optimal for the sellers. We will also consider *competitive equilibrium rules associated to the mechanism.* Under such a *competitive equilibrium rule,* the set of agents is fixed and only the agents' valuations vary. We say that some competitive equilibrium rule is used for a specific market of its domain if we adopt the valuations stated in this market as the agents' true valuations. In this case, the competitive equilibrium rule induces a strategic game. The set of players is the set of agents; the strategies of a player are all possible valuations he can state; the outcome function is given by the competitive equilibrium rule and the true valuations (sincere strategies) are those ones specified by the market. A mechanism is manipulable (or it is not strategy-proof) if there is some competitive equilibrium rule associated to it such that, in some strategic game induced by this rule, the sincere strategy is not a dominant strategy for at least one agent. We must point out that to say that a mechanism is manipulable does not mean that in every strategic game induced by the associated rules there will be some agent for whom honest revelation of his valuations is not the best policy.

When a competitive equilibrium rule is used for a market, questions on incentives facing agents naturally emerge. The first important result in this direction is the Non-Manipulability Theorem due to Demange (1982) and Leonard (1983). These authors proved that:

> *Let M be a buyer-seller market. If the buyer- optimal (respectively, seller-optimal) competitive equilibrium rule is used for M then, in the induced strategic game, truth telling is a dominant strategy for each buyer (respectively, seller)*[4].

Consequently, the mechanism that yields the optimal competitive equilibrium for a given side of the market is non-manipulable by the agents of that side. An immediate corollary of this result is that any competitive equilibrium rule is strategy-proof whenever the set of competitive equilibrium allocations for M is a singleton.

Following this line of research, another important result is the impossibility

[4] Demange and Gale (1985) proved a more general theorem, which can be obtained as a corollary of Theorem 9.23 of Roth and Sotomayor (1990, 1992), due to Sotomayor.

theorem, established by Theorem 7.3 of Roth and Sotomayor (1990,1992):

Any competitive equilibrium mechanism for the class of buyer-seller markets is manipulable.

The proof consists in showing that there is a market with n buyers and one seller such that, under any competitive equilibrium mechanism, some agent has the incentive to misrepresent his/her valuations. We must point out that this result does not asserts that if a competitive equilibrium rule is used for a market which has more than one equilibrium price, then there will be some agent who will be able to increase his/her payoff by misrepresenting his/her valuations.

In their paper of 1985, Demange and Gale present several examples of markets where a competitive equilibrium rule yields the optimal competitive equilibrium for a given side of the market. Then, an agent belonging to the other side can increase his/her true payoff by misrepresenting his/her valuations. These examples prove that if a mechanism produces the optimal competitive equilibrium for a given side of the market then the mechanism is manipulable by some agent belonging to the other side. However, these examples do not prove that if the optimal competitive equilibrium rule for a given side is used for a market which has more than one equilibrium price, then there will be some agent belonging to the other side who has incentive to misrepresent his/her valuations.

It turns out that the belief that the phenomena observed in the particular market used in the proof of the impossibility theorem and in the particular markets of the examples of Demange and Gale (1985) always occur in any market of the domain of the mechanism has supported, along the years, the following *Folk Theorems*, never proved in the literature:

Manipulability Theorem. *Consider the buyer-seller market* M*. Suppose* M *has more than one competitive equilibrium price. If the buyer- optimal (respectively, seller-optimal) competitive equilibrium rule is used for* M*, then there is some seller (respectively, buyer) who can profitably misrepresent his/her valuations, assuming the other agents tell the truth.*

General Impossibility Theorem. *For every buyer-seller market with more than one equilibrium price, there is no competitive equilibrium rule that gives to every agent the incentive to state his/her true valuations.*

Both theorems, together with the non-manipulability theorem, have constituted the foundation of the theory on incentives for the one-to-one continuous matching models.

In the present work we provide the proofs and formal statements of the two Folk Theorems mentioned above, aiming to fill an important gap in the literature of the continuous one-to-one two-sided matching model. Indeed, we are able to prove a more general Manipulability Theorem than the result that the examples of Demange and Gale (1985) suggest, since it is not required that the competitive equilibrium rule produces the optimal competitive equilibrium for one of the sides:

General Manipulability Theorem. *Consider the buyer-seller market M. Suppose some competitive equilibrium rule for M does not yield the buyer-optimal (respectively, seller-optimal) competitive equilibrium when the profile of sincere strategies is selected. Then, under this strategy profile, some buyer (respectively, seller), who is not receiving his/her optimal competitive equilibrium payoff, is not playing his/her best response.*

The conclusion of the General Impossibility Theorem then follows immediately, as corollary. It is also immediate that:

"A competitive equilibrium rule for a given buyer-seller market M is strategy-proof if and only if the set of competitive equilibrium allocations of M is a singleton".

This note is organized as follows. In section 2 we present the cooperative framework and some results, already known, that will be used in section 4. The competitive equilibrium mechanisms and the competitive equilibrium rules are described in section 3. In section 4 we state and prove our main results. The discussion of the general case is presented in section 5.

2-THE COOPERATIVE FRAMEWORK AND PRELIMINARIES

There are a set P with m buyers, $P=\{p_1,p_2,\ldots,p_m\}$, and a set Q with n sellers, $Q=\{q_1,q_2,\ldots,q_n\}$. Each seller owns one indivisible object. We will denote the seller as well as his object with the same notation. Letters j and k will be reserved to index buyers and objects (or sellers), respectively. Each seller q_k values his object at $s_k\geq 0$.

Each buyer p_j values object q_k at $\alpha_{jk} \geq 0$ and wants to buy one object, at most. Thus, if buyer p_j purchases object q_k at price $\pi_k \geq s_k$, her payoff will be $u_j = \alpha_{jk} - \pi_k$ and the payoff of seller q_k will be $v_k = \pi_k - s_k$. The potential gains from trade between j and k will be $u_j + v_k = \alpha_{jk} - s_k$. We will denote by α_j the vector of values α_{jk}'s; the valuation matrix of the buyers and the valuation vector of the sellers will be denoted by α and s respectively. The buyer-seller market is then described by $\boldsymbol{M(\alpha,s) \equiv (P,Q,\alpha,s)}$. When each seller's reservation price is 0 the corresponding model is the well known Assignment Game presented in the book of Roth and Sotomayor (1990), due to Shapley and Shubik(1972).

An object q_k is **acceptable** to buyer p_j if and only if $\alpha_{jk} - s_k \geq 0$. Thus, an object is **not acceptable** to a buyer if there is no feasible price at which the buyer wishes to buy the object. Set $a(s)_{jk} \equiv \alpha_{jk} - s_k$ if $\alpha_{jk} - s_k \geq 0$ and $a(s)_{jk} \equiv 0$, otherwise. We will use the notation $\sum_j$ for the sum over all p_j in P, $\sum_k$ for the sum over all q_k in Q and $\sum_{j,k}$ for the sum over all p_j in P and q_k in Q.

Definition 1. *A **matching** for $M(\alpha,s)$ is a matrix $x = (x_{jk})$ of zeros and ones. A matching x for $M(\alpha,s)$ is **feasible** if it satisfies (a) $\sum_j x_{jk} \leq 1$ $\forall q_k \in Q$, (b) $\sum_k x_{jk} \leq 1$ $\forall p_j \in P$ and (c) if $x_{jk}=1$ then $\alpha_{jk} - s_k \geq 0$.*

Conditions (a) and (b) state, respectively, that a feasible matching assigns an object to one buyer at most and a buyer to one object at most; condition (c) means that the object matched to a buyer is acceptable to her.

If $x_{jk}=0$ for all $q_k \in Q$ (respectively $p_j \in P$), we say that p_j (respectively q_k) is **unmatched**. If $x_{jk}=1$, we say that p_j is matched to q_k or q_k is matched to p_j.

*A matching x for $M(\alpha,s)$ is **optimal** if it is feasible and, for all feasible matchings x', $\sum_{j,k} a(s)_{jk} x_{jk} \geq \sum_{j,k} a(s)_{jk} x'_{jk}$. In addition, if p_j and q_k are both unmatched, then q_k is not acceptable to p_j.*

This concept is illustrated by the following example.

Example. Consider $P=\{p_1,p_2\}$, $Q=\{q_1,q_2\}$, $\alpha_1=(3,3)$, $\alpha_2=(1,2)$ and $s=(3,0)$. Both objects are acceptable to p_1, but q_1 is not acceptable to p_2. The optimal matching for $M(\alpha,s)$ matches q_2 to p_1 but leaves q_1 and p_2 unmatched. ■

Definition 2. *A **feasible outcome*** $(u,v;x)$ *for* $M(\alpha,s)$ *is a pair of vectors* (u,v)*, with* u *in* R^m *and* v *in* R^n*, plus a feasible matching* x *such that* $\sum_j u_j + \sum_k v_k = \sum_{j,k} a(s)_{jk} \cdot x_{jk}$*. If* $(u,v;x)$ *is a feasible outcome then* (u,v) *is called **feasible payoff**.*

If $(u,v;x)$ is a feasible outcome we say (u,v) and x are **compatible** with each other. The notion of stability is given by the following definition:

Definition 3. *The outcome* $(u,v;x)$ *is **stable** for* $M(\alpha,s)$ *if it is feasible and, for all* $(p_j,q_k) \in PxQ$*, (a)* $u_j \geq 0$*,* $v_k \geq 0$ *(individual rationality) and (b)* $u_j + v_k \geq a(s)_{jk}$*.*

If $(u,v;x)$ is stable for $M(\alpha,s)$, we say (u,v) is a stable payoff for $M(\alpha,s)$.

Remark 1. It can be easily proved that if $(u,v;x)$ is stable for $M(\alpha,s)$ then *(c)* $u_j+v_k=a(s)_{jk}$ if $x_{jk}=1$ and *(d)* $u_j=0$ for all unmatched p_j, and $v_k=0$ for all unmatched q_k. On the other hand, every outcome that satisfies (c) and (d) for a feasible matching, is feasible for $M(\alpha,s)$. Then, **an outcome is stable if and only if it satisfies (a)-(d).** ■

For the case where $s=(0,\ldots,0)$, Shapley and Shubik (1972) proved that **the core of** $M(\alpha,s)$ **is non-empty and equals the set of stable payoffs.** They also showed that, for this market, there is a ***P*-optimal stable payoff** such that all buyers (weakly) prefer it to every other stable payoff, and all sellers (weakly) prefer any other stable payoff to it, and there is a ***Q*-optimal stable payoff** with symmetric properties. Clearly, the same results apply to $M(\alpha,s)$ for any reservation price vector s.

A **feasible price vector** π (feasible price, for short) for market $M(\alpha,s)$ is a function from Q to R, such that $\pi_k \equiv \pi(q_k)$ is greater than or equal to s_k. A **feasible allocation** for $M(\alpha,s)$ is a pair (π,x), where π is a feasible price and x is a feasible matching. The payoff vector of the buyers corresponding to a feasible allocation (π,x) is feasibly defined: $u_j=\alpha_{jk}-\pi_k$ if $x_{jk}=1$ and $u_j=0$ if p_j is unmatched.

The **demand set** of buyer p_j at the feasible price π is the set:

$$D(p_j,\pi)=\{q_k\in Q;\ \alpha_{jk}-\pi_k\geq 0 \text{ and } \alpha_{jk}-\pi_k\geq\alpha_{jt}-\pi_t \text{ for all } q_t\in Q\}.$$

Observe that if $q_k\in D(p_j,\pi)$ then $\alpha_{jk}-s_k\geq 0$. Thus, among all the acceptable objects that p_j can acquire at price π, p_j demands those ones which maximize her payoff at the given prices.

Definition 4. *A feasible allocation* (π, x) *for* $M(\alpha,s)$ *is a* ***competitive equilibrium*** *if (a) for all pair* (p_j,q_k) *with* $x_{jk}=1$*, we have that* $q_k\in D(p_j,\pi)$*; (b) if* p_j *is unmatched then* $\alpha_{jk}-\pi_k\leq 0$ $\forall q_k\in Q$ *and (c) if* q_k *is unmatched then* $\pi_k=s_k$.

If (π,x) is a competitive equilibrium for $M(\alpha,s)$, π will be called *equilibrium price vector* or simply *equilibrium price* and x will be called a *competitive matching.* In this case, we say that x is compatible with π and vice versa. Since $\pi_k\geq s_k$ for every q_k, it follows that the corresponding payoff vector $(u,\pi-s)$ is feasible for $M(\alpha,s)$. It will be called *competitive equilibrium payoff.*

Definition 5. *The feasible allocation* (u^*,v_*) *is the* ***P-optimal competitive equilibrium payoff*** *if* $u^*_j\geq u_j$ *for all stable payoff* (u,v)*. The* ***Q- optimal competitive equilibrium payoff*** (u_*,v^*) *is symmetrically defined.*

It is easy to verify that (π,x) *is a competitive equilibrium for* $M(\alpha,s)$ *if and only if the corresponding competitive equilibrium payoff* $(u,\pi\text{-}s)$ *is stable for* $M(\alpha,s)$.

Thus, in $M(\alpha,s)$, the *P-optimal competitive equilibrium payoff* (u^*,v_*) is the P-optimal stable payoff and corresponds to the *minimum equilibrium price* $\pi_*\equiv v_*+s$ (or *P-optimal equilibrium price).* This is the equilibrium price vector that is at least as small in every component as any other equilibrium price vector. Symmetrically, the *Q-optimal competitive equilibrium payoff* (u_*,v^*) is the Q-optimal stable payoff and corresponds to the *maximum competitive equilibrium price* $\pi^*\equiv v^*+s$ (or *Q-optimal competitive equilibrium price).*

It is implied by Proposition 1* below that if x is an optimal matching and π is a competitive equilibrium price for $M(\alpha,s)$, then (π,x) is a competitive equilibrium for $M(\alpha,s)$. Conversely, if (π,x) is a competitive equilibrium for $M(\alpha,s)$, then x is an optimal matching for $M(\alpha,s)$.

The following results will be used in the next sections.

Proposition 1* (Shapley and Shubik, 1972). (*a) If* x *is an optimal matching for market* $M(\alpha,s)$*, then it is compatible with any stable payoff* (u,v) *for* $M(\alpha,s)$*; (b) If* $(u,v;x)$ *is a stable outcome for* $M(\alpha,s)$*, then* x *is an optimal matching for* $M(\alpha,s)$.

Proposition 2* (Demange and Gale, 1985). *Let* $(u,v;x)$ *be a stable payoff for* $M(\alpha,s)$*. Then, if* $u_j>0$ *(respectively* $v_k>0$*),* p_j *(respectively* q_k*) is matched at any optimal matching for* M.

Define $V_{\alpha,s}(P,Q)\equiv\max\sum_{R\times S} a(s)_{jk}\cdot x_{jk}$, with the maximum to be taken over all feasible matchings x for $M(\alpha,s)$.

Proposition 3* (Demange (1982), Leonard (1983)). *Let* (u^*,v_*) *and* (u_*,v^*) *be the P-optimal and the Q-optimal stable payoffs, respectively, for* $M(\alpha,s)$*. For all* q_k *in* Q*, and for all* p_j *in* P,

(a) $v^*_k = V_{\alpha,s}(P,Q) - V_{\alpha,s}(P,Q-\{q_k\})$

(b) $v_{*k} = V_{\alpha,s}(P-\{p_j\},Q) - V_{\alpha,s}(P-\{p_j\},Q-\{q_k\})$ *if* $x_{jk}=1$ *and* $v_{*k}=0$ *if* q_k *is not matched by* x.

(c) $u^*_j = V_{\alpha,s}(P,Q) - V_{\alpha,s}(P-\{p_j\},Q)$

(d) $u_{*j} = V_{\alpha,s}(P,Q-\{q_k\}) - V_{\alpha,s}(P-\{p_j\},Q-\{q_k\})$ *if* $x_{jk}=1$ *and* $u_{*j}=0$ *if* p_j *is not matched by* x.

3. COMPETITIVE EQUILIBRIUM MECHANISMS AND COMPETITIVE EQUILIBRIUM RULES

A competitive equilibrium mechanism for the buyer-seller market is a function that selects some specific competitive equilibrium allocation for every market $M(\alpha',s')$. A competitive equilibrium mechanism can be used to define a competitive equilibrium rule (π,x) for a given market $M(\alpha,s)$, **where π is a price rule and x is a matching rule**. The domain of (π,x) is the set of all possible inputs $M(P,Q,\alpha',s')\equiv M(\alpha',s')$, where the sets P and Q are fixed, and whose "output" is the competitive equilibrium $(\pi(\alpha',s'), x(\alpha',s'))$ for the market $M(\alpha',s')$. The corresponding payoff vector of the buyers is denoted by $u(\alpha',s')$. Thus, $(u(\alpha',s'),\pi(\alpha',s')-s', x(\alpha',s'))$ is the corresponding competitive equilibrium outcome.

If (π,x) always produces the P-optimal (respectively, Q-optimal) competitive equilibrium allocation for every input $M(\alpha',s')$, it will be called P-optimal (respectively, Q-optimal) competitive equilibrium rule.

If a particular competitive equilibrium rule (π,x) is adopted for use in the market $M(\alpha,s)$, it serves as outcome function in the strategic game $\Gamma(\pi,x)$, where the set of players is the set of agents, P and Q; the set of strategies for buyer p is the set of all m-vectors $\alpha'_p\geq 0$; the set of strategies for seller q is the set of all numbers $s'_q\geq 0$. The preferences of the players over the outcomes are determined by their true valuations (α,s). The **true payoffs of buyer p_j and seller q_k** under the allocation $(\pi(\alpha',s');x(\alpha',s'))$, with respect to the market $M(\alpha,s)$, are, respectively:

$U_j(\pi(\alpha',s'); x(\alpha',s'))=\alpha_{jk} - \pi_k(\alpha',s')$ if $x(\alpha',s')_{jk}=1$ and

$U_j(\pi(\alpha',s'); x(\alpha',s'))=0$, if p_j is unmatched at $x(\alpha',s')$.

$V_k(\pi(\alpha',s'); x(\alpha',s'))= \pi_k(\alpha',s')-s_k$ if q_k is matched at $x(\alpha',s')$ and

$V_k(\pi(\alpha',s'); x(\alpha',s'))=0$, otherwise.

4. MAIN RESULTS

As discussed in section 1, the Impossibility Theorem of Roth and Sotomayor (1990,1992) is a very weak result, since the conclusion need not hold for every market of the domain of the mechanism. If the market has only one competitive equilibrium, the Non-Manipulability Theorem, first proved by Demange (1982) and Leonard (1983), implies that it is a dominant strategy for every agent to tell the truth under any competitive equilibrium rule.

This section proves the two Folk Theorems presented in section 1. The first one is a manipulability theorem which confirms that the phenomenon observed in the examples of Demange and Gale (1985) occurs in every buyer-seller market. The second one is a general impossibility theorem that applies to every market with more than one competitive equilibrium price. Both theorems are corollary of a more general result that we call General Manipulability Theorem.

Theorem 1. (General Manipulability Theorem) *Let (π,x) be any competitive equilibrium rule for market $M\equiv M(\alpha,s)$. Let $Y\in\{P,Q\}$. Suppose $(\pi(\alpha,s);\, x(\alpha,s))$ is not a Y-optimal competitive equilibrium for M. Then, some agent in Y is not playing his/her best response when (α,s) is selected. This agent is any $y\in Y$ such that his/her payoff produced by (π,x) is different from his/her payoff under the Y-optimal competitive equilibrium payoff for M.*

Proof. The corresponding competitive equilibrium outcome when agents select (α,s) is $(u',v';x')\equiv(u(\alpha,s),\pi(\alpha,s)\text{-}s;\; x(\alpha,s))$. Let (u^*,v_*) and (u_*,v^*) be the buyer-optimal stable payoff and the seller-optimal stable payoff, respectively, for M. Since $(u',v';x')$ is stable for M, Proposition 1* implies that x' is compatible with (u^*,v_*) and (u_*,v^*).

First case: $Y=P$. By hypothesis $u'\neq u^*$. Let p_j be any buyer such that $u^*_j>u'_j\geq 0$. Then, p_j is matched to some q_k at x', by Proposition 2*, and so $\alpha_{jk}-u^*_j=v_{*k}+s_k$. Also, for some positive λ, $u^*_j>u'_j+\lambda$.

Now define the buyers' strategy profile β as follows: $\beta_{rt}=\alpha_{rt}$, for all $(p_r,q_t)\in PxQ$, with $p_r\neq p_j$; $\beta_{jt}=\alpha_{jt}-(u'_j+\lambda)$ if $\alpha_{jt}-(u'_j+\lambda)\geq 0$ and $\beta_{jt}=0$ otherwise, for all $q_t\in Q$. Define the matrix $b(s)$ by $b(s)_{rt}=\beta_{rt}-s_t$ if $\beta_{rt}-s_t\geq 0$ and $b(s)_{rt}=0$, otherwise.

We are going to show that $U_j(\pi(\beta,s);x(\beta,s))>u'_j$. To have this established we first note that, since $\alpha_{jk}-(u'_j+\lambda)>\alpha_{jk}-u^*_j = v_{*k}+s_k \geq 0$, then $\beta_{jk}=\alpha_{jk}-(u'_j+\lambda)$ and β_{jk}

$>v_{*k}+s_k$, so $b(s)_{jk}>v_{*k}$. Now we claim that p_j is matched under any optimal matching for $M(\beta,s)$. In fact, arguing by contradiction, suppose p_j is unmatched under some optimal matching for $M(\beta,s)$. Then, $V_{\alpha,s}(P\text{-}\{p_j\},Q) = V_{\beta,s}(P,Q) \geq b(s)_{jk} + V_{\beta,s}(P\text{-}\{p_j\},Q\text{-}\{q_k\})>v_{*k}+ V_{\beta,s}(P\text{-}\{p_j\},Q\text{-}\{q_k\})$. Therefore, $v_{*k} < V_{\alpha,s}(P\text{-}\{p_j\},Q) - V_{\alpha,s}(P\text{-}\{p_j\},Q\text{-}\{q_k\})$, which contradicts Proposition 3*-b).

Then, let q_t be the object matched to p_j at $x(\beta,s)$. We have that $\beta_{jt}= \alpha_{jt}\text{-}(u'_j+\lambda)= u_j(\beta,s)+ \pi_t(\beta,s)$, so $\pi_t(\beta,s))\leq\beta_{jt}$. Thus, $U_j(\pi(\beta,s);x(\beta,s))=\alpha_{jt}\text{-}\ \pi_t(\beta,s) \geq \alpha_{jt} \text{-}\beta_{jt}= u'_j+\lambda >u'_j$. Hence, p_j can improve her payoff by deviating from her sincere strategy and the proof of this case is complete.

Second case: $Y=Q$. By hypothesis $v'\neq v^*$. Let q_k be any seller such that $v^*_k>v'_k$. Then, q_k is matched under x'. Also, for some positive λ, $v^*_k>v'_k+ \lambda$.

Now, let s' be a strategy profile for the sellers, where $s'_t=s_t$ for all $q_t\neq q_k$ and $s'_k=v'_k+ \lambda +s_k$. That is, under the profile of strategies (α,s'), q_k replaces his sincere strategy by s'_k and the other players keep theirs. We will show that $V_k(\pi(\alpha,s');x(\alpha,s'))>v'_k$. In fact, first note that $v^*_t+s_t\geq s'_t$ for all q_t implies that (v^*+s,x') is a feasible allocation for $M(\alpha,s')$. Now use the competitivity of v^*+s in $M(\alpha,s)$, and the fact that if q_t is unmatched at x' then $v^*_t+s_t=s_t=s'_t$, to see that (v^*+s,x') is a competitive equilibrium for $M(\alpha,s')$. Since $(v^*_k+s_k)\text{-}s'_k>0$ it follows from Proposition 2* that q_k is matched at any optimal matching for $M(\alpha,s')$, in particular q_k is matched under $x(\alpha,s')$. Hence $V_k(\pi(\alpha,s'),x(\alpha,s'))= \pi_k(\alpha,s')\text{-}s_k\geq s'_k\text{-}s_k=(v'_k+\lambda)>v'_k$, which completes the proof. ■

That is, for any competitive equilibrium rule (π,x), if $(\pi(\alpha,s),x(\alpha,s))$ is not the buyer (respectively, seller)-optimal competitive equilibrium for market $M(\alpha,s)$, then, in the induced game $\Gamma(\pi,x)$, truthful behavior is not a dominant strategy for at least one buyer (respectively, seller). Consequently, when the buyer (respectively, seller)-optimal competitive equilibrium rule is used, some seller (respectively, buyer) can profitably misrepresent his/her valuations, assuming the others tell the truth.

Another immediate consequence is that, for every market $M(\alpha,s)$ with more than one competitive equilibrium, there is no competitive equilibrium rule that gives to every agent the incentive to play his/her sincere strategy. Hence, we have proved the following results already stated in section 1.

Corollary 1. (Folk Manipulability Theorem) *Let* $Y\in\{P,Q\}$*. Suppose that the Y-optimal competitive equilibrium rule for* $M(\alpha,s)$ *is used. If* $M(\alpha,s)$ *has more than one competitive equilibrium price, then some agent who does not belong to* Y *is not playing his/her best response when* (α,s) *is selected .*

Manipulability Theorem. *Suppose* M *has more than one competitive equilibrium price. If the buyer- optimal (respectively, seller-optimal) competitive equilibrium rule is used for* M*, then there is some seller (respectively, buyer) who can profitably misrepresent his/her valuations, assuming the other agents tell the truth.*

Corollary 2. (Folk General Impossibility Theorem) *If* $M(\alpha,s)$ *has more than one competitive equilibrium price, then, under any competitive equilibrium rule there is some agent who has incentive to misrepresent his/her valuations.*

The General Impossibility theorem plus Corollary 2 characterize the domain of the strategy-proof competitive equilibrium rules:

Corollary 3. *A competitive equilibrium rule for a given buyer-seller market* M *is strategy-proof if and only if the set of competitive equilibrium allocations of* M *is a singleton.*

5. CONCLUDING REMARKS

The model treated in the previous sections is "unsymmetrical" in that a seller specifies only one number, his reservation price, while a buyer specifies a *n*-vector, her valuations for each of the objects. There are models, as the job assignment model, where sellers discriminate, specifying different reservation prices to different buyers. In this case, each seller q_k specifies a vector of prices $\pi_k=(\pi_{1k,\ldots,}\ \pi_{mk})$ and each buyer p_j demands the object of seller q_k at prices π if $\alpha_{jk}-\pi_{jk}\geq 0$ and $\alpha_{jk}-\pi_{jk}\geq\alpha_{jt}-\pi_{jt}$ for all $q_t\in Q$. In this model it is more appropriated to think of sellers as being workers who are selling their services to employers.

The job assignment model is a special case of a symmetrical and more general model proposed in Demange and Gale (1985), where the utilities need not be linear. Here the preferences of the agents are given by utility functions: $U_{jk}(x)$ denotes the utility to p_j of being matched with q_k and receiving a monetary payment of x, and $V_{jk}(x)$ denotes the utility to q_k of being matched with p_j and receiving a monetary payment of x. The functions U_{jk} and V_{jk} are continuous and strictly increasing from $\boldsymbol{R}$ onto $\boldsymbol{R}$. It is also assumed that for each p_j and q_k the utility of being unmatched is given by some numbers r_j and z_k. The notion of stability is the usual one.

The outcome $(u,v;x)$ *is stable if for all* $(p_j,q_k)\in PxQ$ *we have that (a)* $u_j\geq r_j$, $v_k\geq z_k$; *(b)* $U_{jk}^{-1}(u_j) + V_{jk}^{-1}(v_k) \geq 0$; *(c)* $U_{jk}^{-1}(u_j) + V_{jk}^{-1}(v_k)=0$ *if* $x_{jk}=1$ *and (d)* $u_j=0$ *if* p_j *is unmatched at* x *and* $v_k=0$ *if* q_k *is unmatched at* x.

The buyer-seller market we have treated here is the special case in which all reservation payoffs, r_j and z_k, equal 0 and $U_{jk}(-x)=\alpha_{jk}-x$ and $V_{jk}(x)=x-s_k$. In the case in which the sellers discriminate the buyers we have that $V_{jk}(x)=x-s_{jk}$.

A competitive approach for the model of Demange and Gale can be obtained here. If we think of the agents as being buyers and sellers, the price that q_k should sell his object to buyer p_j in order to get the payoff v_k is $\rho_{jk}=V_{jk}^{-1}(v_k)$.

Let ρ_k denote the array of prices ρ_{jk} and let ρ denote the n-tuple $(\rho_1,\ldots, \rho_n)$. The feasibility condition requires that $V_{jk}(\rho_{jk})\geq z_k$ for all $q_k\in Q$. Buyer p_j would demand the object q_k at prices ρ if $U_{jk}(-\rho_{jk})\geq r_j$ and $U_{jk}(-\rho_{jk})\geq U_{jt}(-\rho_{jt})$ for all $q_t\in Q$. The payoff of buyer p_j if she purchased the object q_k for ρ_{jk} would be $u_j\equiv U_{jk}(-\rho_{jk})$ and the payoff of seller q_k would be $v_k\equiv V_{jk}(\rho_{jk})$. At a competitive equilibrium (ρ,y) every seller is indifferent between any two buyers. That is, although a seller may have two different prices for two different sellers, he gets the same payoff with both of them. Clearly, (ρ,y) is a competitive equilibrium if and only if $(u,v;y)$ is a stable outcome. We say that $\rho^*=(\rho^*_1,\ldots, \rho^*_n)$ is the seller-optimal equilibrium price if the corresponding payoff (u_*,v^*) is the seller-optimal stable payoff. The buyer-optimal equilibrium price $\rho_*=(\rho_{*1},\ldots, \rho_{*n})$ is symmetrically defined. A competitive equilibrium rule is a function that for any stated preferences $(U,r;V,z)$ produces a competitive equilibrium allocation for the market $M=(P,Q, U,r;V,z)$.

How would this greater generality affect our results? Actually, nothing needs to be changed in the statements of the theorems. In fact, our results hold for the general continuous model. The only difference between the two models would be that in the linear case the reservation payoffs are always 0. This causes the agents to restrict their strategies to their valuations. In the new model an agent can change his/her utility or reservation payoff or both. Thus, the proof of the General Manipulability Theorem should follow the lines of the second part of the proof presented in the previous section, with the necessary adaptation. Instead of the reservation price s_k, agent q_k would change his reservation payoff z_k by stating $z'_k = v'_k + \lambda$. The linearity of the continuous functions is not used, so the arguments apply to the general competitive model. The formal proof is given below.

We will use the following propositions, which are implied by Proposition 9.11 and Theorem 9.8, respectively, from Roth and Sotomayor (1990,1992), due to Demange and Gale (1985).

Proposition 4*. *Let* (u_*,v^*) *and* (u'_*,v'^*) *be the seller-optimal stable payoffs for* $M=(P,Q,U,r;V,z)$ *and* $M'=(P,Q,U,r;V,z')$, *respectively. If* $z'\geq z$, *then,* $v'^*_k \geq v^*_k$ *for all* $q_k \in Q$.

Proposition 5*. *Let* $(u,v;x)$ *and* $(u',v';x')$ *be stable outcomes. If* $v_k > z_k$ *then* q_k *is matched under* x *and* x'.

Proof of Theorem 1 for the general competitive model. Let $M=(P,Q,U,r;V,z)$. Let $(u,v;y)$ be the stable outcome corresponding to the competitive equilibrium $\pi(\theta)$ produced when agents select $\theta=(U,r;V,z)$. Let (u_*,v^*) be the seller-optimal stable payoff for M and let π^* be the seller-optimal equilibrium price. Let x^* be some compatible optimal matching.

We will prove the result for one of the sides. The proof for the other side follows dually. Then, suppose $Y=Q$. By hypothesis $\pi(\theta)\neq\pi^*$, so $v\neq v^*$ (use that the functions V_{jk} are strictly increasing). Let q_k be any seller such that $v^*_k > v_k \geq z_k$. Then, q_k is matched under x^*. Also, for some positive λ, $v^*_k > v_k + \lambda$.

Now, let (V,z') be a strategy profile for the sellers, where $z'_t = z_t$ for all $q_t \neq q_k$ and $z'_k = v_k + \lambda$. That is, under the profile of strategies $\theta'=(U,r, V,z')$, q_k replaces his sincere strategy (V_k,z_k) by (V_k,z'_k) and the other players keep theirs. We will show that the true payoff of q_k under $(\pi(\theta'),x(\theta'))$ is greater than v_k. To see that, first note that by

Proposition 4*, $v'^*_k \geq v^*_k > z'_k$, where (u'_*, v'^*) is the seller-optimal stable payoff for $M(\theta')$. Then, $v'^*_k > z'_k$, so, by Proposition 5*, q_k is matched at any matching compatible with a stable payoff for M', in particular q_k is matched under $x(\theta')$. Hence his true payoff under $(\pi(\theta'), x(\theta'))$ is $v_k(\theta') \geq z'_k = (v_k + \lambda) > v_k$, which completes the proof. ■

Analogous results to the ones proved here were proved for the Marriage market (Roth and Sotomayor, 1990 and 1992) and for the College Admission market (Sotomayor, 2011).

Demange and Gale (1985) analyze the strategic equilibrium of the game induced by the mechanism that produces the buyer-optimal stable payoff when the buyers always play their sincere strategies and the sellers keep fixed their utility function and only manipulate their reservation prices. The analysis of the case in which the mechanism produces any stable payoff and there is no restriction on the strategies selected by the agents is an issue which we intend to approach in future investigations.

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