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Labor Time Shared in the Assignment Game Lending New Insights to the Theory of Two-Sided Marching Markets

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Keywords: stable allocations, core, competitive equilibrium allocations, feasible deviation

JEL Codes: C78, D78

LABOR TIME SHARED IN THE ASSIGNMENT GAME LENDING NEW INSIGHTS TO THE THEORY OF TWO-SIDED MATCHING MARKETS

by

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ABSTRACT

We consider two two-sided matching markets, where every agent has an amount of units of a divisible good to be distributed among the partnerships he forms and exchanged for money. Both markets have the same sets of feasible allocations but operate under distinct rules. However they are indistinguishable under their representation in the characteristic function form. The adequate and proposed mathematical model provides the foundation to characterize the cooperative equilibrium concept in each market. Setwise-stability is then shown not to be the general definition of stability. The connection between the cooperative structures of these markets and between the cooperative and competitive structures of each market is established, by focusing on the algebraic structure of the core, the set of cooperative equilibrium allocations and the set of competitive equilibrium allocations. The results obtained and the methodology used in their proofs provide new and useful insights to the theory of two-sided matching markets.

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INTRODUCTION

Two-sided matching markets where the structure of preferences is given by utility functions that are real and continuous in the money variable, denominated continuous two-sided matching models, have been particularly useful to model environments that can be treated cooperatively and competitively.[2] In these markets the cooperative and the competitive structures can be analyzed altogether and compared to each other, which makes possible that the natural solution concepts – core, cooperative equilibrium and competitive equilibrium – be characterized and correlated.

The characterizations of the cooperative and the competitive equilibrium concepts and the connection between them are of great interest to economists and game theorists in the extent that they contribute to a better understanding of the fundamental similarities and differences between the cooperative and competitive structures of the market under study. This line of research has been developed since Shapley and Shubik (1962), through the continuous two-sided matching models that have been proposed.[3] The purpose of this work is to give continuity to this line of research. This is done through the analysis of two new continuous matching markets which are endowed with distinct market rules, but which are indistinguishable under their representation in the characteristic function form. The central interest of this work is to determine how these rules affect the cooperative structure and the mathematical modeling of these markets, creating distinct algebraic configurations for the respective solution sets. Along this paper we show that these markets present some peculiarities in their cooperative and competitive structures, not observed in the continuous matching models previously presented in the literature.

Historically, the main characteristic of the one-to-one and many-to-one continuous matching models that have been studied in the literature is that the agents' payoffs are one-dimensional, so they can be interpreted as resulting from negotiations in block (Shapley and Shubik (1972), Demange and Gale (1985), Kelso and Crawford (1982), Gul and Stacchetti (1999,2000), Sotomayor (2002)). The agents only care about their total payoffs. This assumption together with the continuity of the utility functions cause

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[2] As an example see Sotomayor (2007).

[3] An overview on the one-to-one continuous matching models can be found in Roth and Sotomayor (1990, 1992) .

the coincidence of the three solution sets, the core, the set o cooperative allocations and the set o competitive equilibrium allocations.

The first continuous many-to-many matching model formally presented in the literature was introduced in Sotomayor (1992) and called multiple-partners assignment game.[4] Since then it has been widely studied in several papers (Sotomayor (1999-a, 2007, 2009), Fagebaume, Gale and Sotomayor (2010)).

The multiple partners assignment game is obtained by incorporating quotas into the assignment game of Shapley and Shubik (1972). The quota of an agent [5] represents the maximum number of partnerships he can form. The agents can be interpreted as being buyers and sellers of indivisible goods. *A buyer cannot acquire more than one item from the same seller.* For each buyer p_i and seller q_j there is a nonnegative number a_{ij} representing the maximum amount of money buyer p_i would consider paying for one of the identical objects owned by seller q_j. This game is also adequate to model labor markets of heterogeneous firms and workers. In this case, the numbers a_{ij} represent the productivity of worker q_j in firm p_i.

The main assumption of this model is that agents' utilities are additively separable. This assumption makes possible that each agent receives a multi-dimensional payoff, one individual payoff corresponding to each trade performed. (Thus, if say, agent p forms three partnerships then he gets a three-dimensional payoff). The negotiations are then pairwise and independent. This independence implies that an agent's individual payoff in a given partnership is not affected if this agent or his partner breaks some of his agreements in other partnerships or adds new agreements to the pool, or leaves some of his partners.

The multi-dimensionality of the individual payoffs in both sides of the market permits to deal with cooperative and competitive structures more game theoretically interesting than those provided by markets where the payoffs are one-dimensional or where the multiple-partnerships are restricted to only one of the sides of the market. In fact, the equivalence between the three solution sets mentioned above is lost in the multiple-partners assignment game. In this model, the concept of cooperative equilibrium for matching markets was viewed, by the first time, as a new concept,

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[4] Crawford and Knoer (1981) study a version of the assignment game of Shapley and Shubik (1972). The authors mention that their model can be extended to a many-to-many assignment game by introducing quotas into their one-to-one matching model. However they focus on the core concept, which characterizes the concept of stability in the one-to-one case, but not in the many-to-many case.

[5] For simplification, we will refer to an agent, a buyer and a seller as "he".

different from the core concept: The core of two-sided many-to-many matching markets contains the set of cooperative equilibrium allocations and may be different from this set (Sotomayor, 1992)[6]. Another result of the multiple partners assignment model is that the set of cooperative equilibrium allocations contains the set of competitive equilibrium allocations and may be different from this set (Sotomayor, 2007).

The results reviewed above are illustrated in the following example.

EXAMPLE 1. There are two firms p_1 and p_2 and two workers q_1 and q_2. Both firms can hire both workers; worker q_1 can take only one job and worker q_2 can take both jobs. The first row of matrix a is (3,2) and the second row is (3,3). Consider the allocation where p_1 hires both workers, gets 1 with each one of them and pays 2 to q_1 and 1 to q_2; p_2 hires only worker q_2 at salary 2 and obtains the net profit of 1. Now observe that p_2 and q_1 could increase their gains by working together. Since the trades are independent, p_2 could keep its partnership with q_2. Then, for example, if p_2 pays 2.5 to q_1 it will get the total payoff of 1.5 instead of 1. Thus, in a cooperative environment, where agents can freely communicate with each other, we cannot expect to observe this allocation, because q_1 will not accept to receive 2 from p_1, since he knows that p_2 can pay him more than 2. Therefore, this allocation cannot be regarded as a cooperative equilibrium. It is unstable. However, this allocation is in the core, since it is not blocked by any coalition. In fact, agents p_2 and q_1 don't block the allocation because they are not able to increase their total payoffs by interacting only among themselves. They need more than 3 and they are able to get 3, at most. Buyer p_2 needs to keep his partnership with q_2, but coalition $\{p_2, q_1, q_2\}$ needs more than 6 to block the allocation and it is able to get 6, at most. On the other hand, the four agents only are able to get 8, at most, which they already have, so they do not form a blocking coalition.

If, instead of paying a salary of 2 to q_1, p_1 pays him 3, the new allocation is stable (and so it is in the core), since p_2 cannot offer to q_1 more than 3. However this allocation is not a competitive equilibrium allocation because buyer p_2 demands q_2 at price 1, not at price 2. ∎

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[6] Later, in Sotomayor (1999-b), the concept of cooperative equilibrium for the discrete many-to-many matching models with substitutable preferences was characterized as the concept of setwise-stability introduced in that paper. Setwise-stability was proved to be distinct from the concept of core and of strong core and stronger than this concept plus pairwise-stability.

Another way to compare the three solution sets is to focus on their algebraic structure. In Sotomayor (1999-a), it is proved for the multiple-partners assignment model that the core is not a lattice and that the set of cooperative equilibrium payoffs is a complete lattice under two partial order relations, not defined by the preferences of the agents. In spite of this, the extreme points of the lattices of cooperative equilibrium payoffs are the optimal stable payoffs for each side of the market.

The algebraic structure of the set of competitive equilibrium allocations is treated in Sotomayor (2007), where it is proved that the set of competitive equilibrium allocations can be obtained by shrinking the set of cooperative equilibrium allocations through a function f which preserves the order. This function maps a cooperative equilibrium allocation to a feasible allocation obtained by reducing, to their minimum, the individual payoffs of each seller and by defining feasibly the payoffs of the buyers. The resulting allocation is still a cooperative equilibrium. It is also proved that the competitive equilibrium allocations are characterized as the cooperative equilibrium allocations in which the sellers do not discriminate the buyers. This characterization permits to identify the competitive equilibrium allocations with the fixed points of f. Then, Tarski's fixed point theorem (Tarski, 1955) implies that the set of competitive equilibrium allocations is a non-empty sub-lattice of the lattice of cooperative equilibrium allocations, establishing a link between the cooperative and competitive structures of the multiple partners assignment game.

In the models proposed here, we modify the rules of the multiple partners assignment game: *a buyer is allowed to acquire more than one unit of the good of a seller*. Indeed, it is not required that the good is indivisible or that the quotas of the agents are integer numbers. Then, a buyer is also allowed to acquire less than one unit of the good of a seller.

The basic model is called time-sharing assignment game. It can be obtained by incorporating time into the assignment game of Shapley and Shubik. Thus, there are two finite and disjoint sets of agents, P and Q. Every agent has a quota, given by a positive number, representing the amount of units of a divisible good (e.g., labor time), which are to be distributed in any way he likes among the partnerships he forms. For each pair (p,q) there is a non-negative number a_{pq}, representing the amount of income which p and q can generate if these players contribute one unit of labor time (u.l.t. for short) to this partnership.

We assume that the whole agreement between p and q is broken once its terms with respect to the division of the income a_{pq} are changed. As for the terms on the contribution of labor time we distinguish two kinds of agreements. Under a r*igid agreement,* if p or q breaks the agreement regarding the amount of labor, then the whole agreement, including the division of the income, must be nullified. The market where all agreements are rigid is called *time-sharing assignment game with rigid agreements* or *rigid market*, for short. A flexible agreement between p and q allows either agent to decrease the number of u.l.t. he contributes to the partnership without breaking the agreement corresponding to the division of the income per u.l.t.. Therefore, any of the two agents is allowed to transfer part of his common labor time to some other current partnerships or to some new partnerships. The market where all agreements are flexible is called *time-sharing assignment game with flexible agreements* or *flexible market*, for short. We will be interested on the rigid market and on the flexible market. The flexible market is an auxiliary market which works as an analytical tool for the proofs of our results.[7]

However, the feasible allocations of the rigid and flexible markets do not inform if the agreements inside each partnership are rigid or flexible. Consequently, the distinct rules of these markets yield the same set of feasible allocations, and so the two markets are indistinguishable under their representations in the characteristic function form. In the text it is presented an example (Example 2.2.1) where some deviation is allowed in the flexible market but it is not so in the rigid market. This causes some given allocation to be a cooperative equilibrium allocation for the rigid market and not to be so for the flexible market. Thus, the use of the characteristic function in modeling the rules of the rigid and the flexible markets seems to be inappropriate for the purpose of observing cooperative equilibrium allocations.[8]

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[7] This kind of technique was also employed by Gale and Shapley (1962). These authors defined the marriage model to be used as a tool to prove the existence of stable matchings for the college admission market.

[8] This fact motivated the studies developed in Sotomayor (2012), whose main proposal is a mathematical model, called deviating function form, which allows defining a general concept of stability for all cooperative decision situations in which agents form coalitions and freely interact inside each coalition that is formed, by acting according to some established rules.

The appropriate and proposed mathematical model is the *deviation function form* introduced in Sotomayor (2012). Once this foundation is laid out, it becomes possible to identify the cooperative equilibrium allocations in each market.

A special case of this model, in which the goods are indivisible, was introduced in Thompson (1980). The author uses a non-standard definition of the core however: no agent discriminates any other agent in equilibrium (i.e., each seller sells all of his units for the same price and each buyer gets the same individual payoff in all of his trades). The model with the standard definition of the core was studied in Sotomayor (2002), where Thompson's assumption is relaxed and the (continuous) time-sharing assignment game with one-dimensional payoffs is analyzed. Baiou and Balinski (2002) introduced the discrete case (ordinal preferences) under flexible agreements. The discrete flexible market was also studied in Alkan and Gale (2003) under more complex ordinal preferences.

To date, however, no one has considered the two (continuous) time-sharing assignment games with multi-dimensional payoffs proposed here. They are genuinely new models. Their analysis approaches some of the main issues that arise in the study of matching problems, namely the characterization and existence of the core allocations, of the cooperative equilibrium allocations and of the competitive equilibrium allocations, as well as the correlation between the cooperative and the competitive structures, by focusing on the algebraic structure of the corresponding solution sets, lending new insights to the theory of two-sided matching markets.

Since Gale and Shapley (1962) the cooperative equilibrium allocations in matching markets are called stable allocations. The general idea proposed in Sotomayor (2012), which is extended to non-matching games, is that: *The allocation σ is stable if there is no coalition S of players who can profitably deviate from σ by acting according to the rules of the game. Coalition S is called a deviating coalition.*

This idea contrasts with that of the core: *The allocation σ is in the core if there is no coalition S of players who can profitably deviate from the given allocation by interacting only among themselves. Coalition S is called a blocking coalition.*

Then, it should be specified by the rules of the game if a coalition can or cannot do more than to merely interact among themselves.

In order to characterize the cooperative equilibrium allocations in the rigid and in the flexible markets, we first define in section 2.3 two special cooperative solution concepts - setwise-stability and strong-stability - via two kinds of domination relations. Then, by using the representation of these two markets under the deviation function form defined in section 2.2, we characterize the cooperative equilibrium allocations for the rigid market as the setwise-stable allocations, and the cooperative equilibrium allocations for the flexible market as the strongly-stable allocations. The definition of these concepts implies that the core contains the set of setwise-stable allocations, which contains the set of strongly stable allocations. Examples presented in the text (sections 2.2 and 2.3) show that these inclusions may be strict.

The existence proof of cooperative equilibrium allocations provides a new technique, not used yet in the continuous matching models but very subtle: *competitive equilibrium allocations are stable and always exist.*

The stability of the competitive equilibrium allocations for the multiple partners assignment game was obtained in Sotomayor (2007) through the characterization of the competitive equilibrium allocations as "the stable allocations which do not discriminate the buyers". It turns out that, as it is viewed here, it is not the property of stability plus buyer-non-discrimination that characterizes the competitive equilibrium allocations in our models. In these models, the characterization of the competitive equilibrium allocations requires the additional property of being strongly-stable. *The competitive equilibrium allocations are the strongly-stable allocations which do not discriminate the buyers.* An example in section 4 illustrates a situation in which a stable allocation for the rigid market, which does not discriminate the buyers, is not a competitive equilibrium. Nevertheless, such inconsistency with the multiple-partners assignment model is indeed apparent. As consequence of our results, an allocation is stable for that model if and only if it has the strong-stability property, conveniently adapted.

The main consequence of such characterization is that the cooperative structure of the flexible market can then be viewed as a bridge between the competitive and cooperative structures of the rigid model: *the competitive equilibrium allocations are the stable allocations of the flexible market which do not discriminate the buyers*. As a corollary, *the competitive equilibrium allocations are stable in the rigid market and they are in the core.*

The property of strong-stability of the competitive equilibrium allocations naturally emerges from a non-obvious characterization of the set of strongly-stable allocations provided here. To understand the idea behind this characterization, consider the prices specified by a given feasible allocation. To fix ideas consider a feasible allocation where, say, buyer p_1, with a quota of 5, acquires 3 units of the good of q_1 for \$5 per unit, and acquires 2 units of the good of q_2 for \$2 per unit. Assume for simplicity that the goods are indivisible. Given the prices specified by the feasible allocation, the problem faced by p_1 is to choose a bundle with five units which gives him the highest total payoff. However there are some rules. If he wants to acquire some units of q_1 or q_2, he must pay \$5 for the first three units of q_1 and 2\$ for the first two units of q_2. It turns out that, if the allocation is strongly-stable then the set of units assigned to p_1 by the feasible allocation is in the set of bundles which might be selected by p_1. The allocation is strongly stable if and only if every buyer is assigned a bundle of his set of selected bundles. Then, under a strongly-stable allocation in which no seller discriminates any buyer, every buyer, taking as given the prices of the u.l.t. supplied by the sellers, is maximizing his total surplus. This is precisely how the concept of competitive equilibrium allocation is defined!

Having proved that competitive equilibrium allocations are strongly-stable and do not discriminate the buyers, we address the existence problem. We show that the dual allocations, naturally derived from the dual solutions of the transportation problem conveniently defined, are the strongly-stable allocations in which no agent is discriminated, so they are competitive and always exist. Since every dual solution is compatible with any optimal assignment of the corresponding linear programming problem, it follows that every such dual allocation is compatible with any optimal labor time allocation. (This property is not, in general, shared with the allocations of the other sets). Then we get a stronger result: *the core, the set of stable allocations for both markets and the set of competitive equilibrium allocations, compatible with any optimal labor time allocation, are also non-empty***.**

In the Appendix II we define a demand correspondence which applies to the time sharing assignment game and to the assignment game. The competitive equilibrium for the resulting competitive market is called *competitive equilibrium under non-discriminatory demand.* The notion of demand correspondence used in the text is referred there as *discriminatory demand.* We show that the set of competitive equilibrium under non-discriminatory demand coincides with the set of dual allocations and, under the rules of the assignment game, the two sets of competitive equilibrium allocations coincide. This suggests that the equivalence between the dual allocations and the competitive equilibrium allocations under discriminatory demand is less robust to the introduction of time into the assignment game than the equivalence between the set of dual allocations and the set of competitive equilibrium allocations under non-discriminatory demand.

In section 5 we go beyond simple considerations provided by the set inclusion relation, and we also examine the connection between these solution sets, by focusing on their algebraic structure. As in the multiple-partners assignment game, the core is not a lattice. Nevertheless, the changes in the rules of the multiple partners assignment game affect the lattice property of the set of stable allocations and of the set of competitive equilibrium allocations. None of these sets is a lattice in both markets.

If we restrict the allocations to the same labor time allocation, we can define two partial order relations defined in the text, ≥P (defined by the P-agents) and ≥Q (defined by the Q-agents). The agents can compare their vectors of individual payoffs componentwise. Even in this case the core and the set of stable allocations for the rigid market are not always lattices. In spite of this, Example 5.1 in the text presents a situation in which the *P*-optimal core allocation and the *P*-optimal setwise-stable allocation exist. However, that polarization of interests between the two sides of the market, presented in the multiple-partners assignment game when one compares the two optimal stable allocations, is not observed in the rigid market: The best setwise-stable allocation for the *P*-agents is not the worst setwise-stable allocation for the *Q*-agents. Indeed, there is no setwise-stable money allocation that is the worst setwise-stable money allocation for the *Q*-agents in that model.

The central result of section 5 is that, *given any optimal labor time allocation x, the sets of strongly stable allocations and of competitive equilibrium allocations, compatible with x, are non-empty complete lattices*. However, the changes in the rules of the multiple partners assignment game affect the algebraic structure of the set of stable allocations of the two markets in two different ways. In the rigid market, as mentioned above, while the set of stable allocations is not a lattice (even if restricted to the same labor time allocation), in the flexible market, this set splits into several non-empty lattices, each one corresponding to an optimal labor time allocation. Such a structure does not guarantee the existence of the optimal stable allocations for each side of the flexible market, however, because the corresponding extreme points of these lattices may generate different total payoffs to the agents.

A similar effect is caused on the set of competitive equilibrium allocations: this set also has a non-standard algebraic structure of a union of complete lattices. Each of these lattices is a sub-lattice of the corresponding lattice of strongly-stable allocations.

Nevertheless, in contrast with the set of strongly-stable allocations, each two of the lattices of competitive equilibrium allocations are in one-to-one correspondence through a function which preserves the total payoff of the agents. The extreme points of the first lattice are mapped to the corresponding extreme points of the other lattice. Thus, all suprema are P-optimal competitive equilibrium allocations if we use the partial order $\geq_P$, and they are Q-optimal competitive equilibrium allocations if we use the partial order $\geq_Q$. Hence, *there exist the P-optimal and the Q-optimal competitive equilibrium allocations.*

The existence of the optimal competitive equilibrium allocations for each side of the market has some implication of technical order, since it makes treatable, for our models, several problems of economic interest, whose solution involves the choice of only one convenient feasible allocation. This is, for example, the case of the comparative static problems caused by the entrance of new agents in the market and the stable allocation problems in a centralized market (we provide more details for these problems in section 6).

We also show that the lattice property holds for the whole set of competitive equilibrium price vectors under the partial order $\geq Q$ defined by the sellers. Indeed, this set is a complete lattice under the partial order defined by the total payoffs of the sellers. The lattice property, under $\geq P$ and $\geq Q$, also holds for the whole set of competitive equilibrium allocations under non-discriminatory demand defined in the Appendix II.

This work is organized as follows. In section 2 we present the cooperative structure of the rigid and flexible markets. In sub-section 2.1 we describe the time-sharing assignment market and give the preliminary definitions. In sub-section 2.2 we present an example which illustrates that the nature of the agreements is a relevant information for the problem of modeling the rules of the game. The primitives of the mathematical model in the deviation function form are then presented. Sub-section 2.3 defines the concept of cooperative equilibrium translated to the rigid and flexible markets. It defines the solution concepts of setwise-stability and strong-stability and establishes a set inclusion relation between the corresponding solution sets and the core. Some examples illustrate the strictness of this set inclusion relation. Then the cooperative equilibrium allocations are characterized as the setwise-stable allocations in the rigid market and as the strong-stable allocation in the flexible market. Two results related to the core are also proved. Section 3 presents the competitive framework. Sub-section 3.1 describes the competitive market and sub-section 3.2 defines the competitive equilibrium concept. Section 4 addresses the existence problem. Section 5 concerns the algebraic-structure of the core, the set of cooperative equilibrium allocations in both markets and the set of competitive equilibrium allocations. It is also discussed the lattice property of the set of competitive equilibrium prices under two partial order relations. Final remarks and related work are presented in the last section. Appendix I provides the proofs of the results. Appendix II discusses the competitive market with non-discriminatory demands.

2. COOPERATIVE STRUCTURE OF THE TIME-SHARING ASSIGNMENT GAME

2.1 THE MATCHING MARKET AND SOME PRELIMINARIES

There are two finite and disjoint sets of agents, P with m elements and Q with n elements, which we may think of as a set of buyers and a set of sellers, or a set of firms and a set of workers. We will describe the market in terms of buyers and sellers, who will sometimes be called P-agents and Q-agents, respectively. Generic agents of P and Q will be denoted by p and q, respectively.

Dummy-agents, denoted by 0, will be included in both sides of the market for technical convenience. Set $P^*\equiv P\text{-}\{0\}$ and $Q^*\equiv Q\text{-}\{0\}$. Each agent has a quota of units of labor time (e.g. man-hours) at his disposal, which he can distribute among the partnerships he forms in any way he likes. Each seller q supplies a quota of $s(q)\in R^+$ units of labor time (u.l.t. for short) and each buyer p cannot acquire more than his quota of $r(p)\in R^+$ u.l.t.. The quota $r(0)$ of the dummy P-agent is equal to $\sum_{q\in Q} s(q)$ and the quota $s(0)$ of the dummy Q-agent is equal to $\sum_{p\in P} r(p)$. We will assume that the reservation price of one u.l.t. is zero for all sellers. For each pair $(p,q)\in P\text{x}Q$ there is a nonnegative number a_{pq}, which is to be split between the partners in any way they agree. The number a_{pq} can be interpreted as the maximum amount of money buyer p would consider paying for one unit of labor time supplied by seller q. Then, a_{pq} is the gain from trade when one u.l.t. of seller q is sold to buyer p. Thus, if seller q sells one u.l.t. to buyer p at price w_{pq} then p will get the individual payoff of $u_{pq}=a_{pq} - w_{pq}$ and q will receive w_{pq}. The matrix of numbers a_{pq}'s will be denoted by a, with $a_{p0}=a_{0q}=0$ for all $(p,q)\in P\text{x}Q$. The agents who are not dummies will some times be called *real agents.* We will denote by r and s, respectively, the sets of numbers $r(p)$′s and $s(q)$′s.

The negotiations inside a partnership $\{p,q\}$ must specify:

(1) how the partners should divide the income a_{pq} they get per u.l.t. (so the agents' payoffs are multi-dimensional , one individual payoff for each trade) and

(2) how much labor they should contribute to the partnership.

Then an allocation specifies a money allocation and a labor time allocation, which are indexed according to the partnerships formed.

We will assume that for a partnership (p,q) to be active both members must contribute the same positive amount of units of labor time and each agent should receive equal individual payoff per each u.l.t. he contributes to the partnership. This assumption is natural under our buyer-seller market interpretation: if a trade between a buyer and a seller is performed then the number of units of labor time acquired by the buyer is equal to the number of units of labor time sold by the seller. Furthermore, all these u.l.t. are sold to the buyer for the same price and so the buyer gets the same individual payoffs with all of them. This market will be called *time-sharing assignment game.*

For some purposes (e.g., for observing cooperative equilibria) the rules of the market should also specify the kind of agreement concerning the amount of labor time which is to be contributed to the partnership by its members. We consider two types of agreements on the contribution of the labor time. Under a r*igid agreement,* if p or q breaks the agreement regarding the amount of labor, then the whole agreement, including the division of the income, must be nullified. A flexible agreement between p and q allows either agent to decrease the number of u.l.t. he contributes to the partnership without breaking the agreement corresponding to the division of the income per u.l.t.. Therefore, any of the two agents is allowed to transfer part of his common labor time to some other current partnerships or to some new partnerships.

The market where all agreements are rigid is called time-sharing assignment game with rigid agreements or rigid market, for short. The market where all agreements are flexible is called time-sharing assignment game with flexible agreements or flexible market, for short. We will be interested on the rigid market and on the flexible market.

The players seek to form sets of partnerships to distribute all their labor time. The obvious condition for feasibility is that all money generated by a partnership per u.l.t. is distributed among its members. Formally,

Definition 2.1.1 *A **labor time allocation** is a real matrix* $x=(x_{pq})_{(p,q)\in PxQ}$. *The labor time allocation x is feasible if*

(a) $\sum_{q\in Q} x_{pq}=r(p)$ *for all* $p\in P^*$;

(b) $\sum_{p\in P} x_{pq}=s(q)$ *for all* $q\in Q^*$.

(c) $x_{pq}\geq 0$ *for all pairs* $(p,q)\in P\text{x}Q$.

The number x_{pq} (non-necessarily integer) may be interpreted as the amount of labor time p and q work together. Note that *(a)* is an equation, not an inequality, because p can always contribute left over labor time to the partnership *(p,0)*. Similar observation applies to *(b)*.

A feasible labor time allocation x is *optimal* if

(d) $\sum_{(p,q)\in PxQ} a_{pq} x_{pq} \geq \sum_{(p,q)\in PxQ} a_{pq} x'_{pq}$, *for all feasible labor time allocations* x'.

NOTATION: For the labor time allocation x, set $C(x)\equiv\{(p,q)\in P\text{x}Q;\ x_{pq}>0\}$. If $(p,q)\in C(x)$ we say that (p,q) is *active at x* (or simply *active*, for short, when there is no confusion). We also say that p is matched to q or q is matched to p at x. For the labor time allocation x and $(p,q)\in PxQ$, set $B(p,x)\equiv\{q'\in Q;\ (p,q')\in C(x)\}$ and $B(q,x)\equiv\{p'\in P;\ (p',q)\in C(x)\}$.

Definition 2.1.2. *Given a labor time allocation x, a **money allocation** (u,w) corresponding to x is a pair of non-negative real functions on C(x). It is feasible if x is feasible and*

(e) $u_{pq}+w_{pq}= a_{pq}$ for all $(p,q)\in C(x)$.

We say that (u,w) is compatible with x and vice-versa. The triple (u,w;x) is called a feasible allocation and we also say that it is compatible with x.

That is, *(u,w;x)* is a feasible allocation if it satisfies *(a), (b), (c)* and *(e)*. Condition *(e)* clearly implies that $u_{p0}=w_{0q}=0$ if the corresponding partnerships are active. Observe that u_{pq} is not defined if $x_{pq}=0$. If $x_{pq}>0$ we can also say that p and q are matched to each other under *(u,w;x)*.

It is worth to point out that Definition 2.1.2 does not take into consideration the nature of the agreements, so both markets have the same set of feasible allocations.

NOTATION: (i)We will denote by Σ the set of all feasible allocations. A player compares two feasible allocations by comparing his total payoff in each allocation. The p's total payoff and the q's total payoff generated by $(u,w;x)$ are given, respectively, by: $U_p=\sum_{q\in B(p,x)} u_{pq}x_{pq}$ and $W_q=\sum_{p\in B(q,x)} w_{pq}x_{pq}$. (ii) For every $p\in P$ and $q\in Q$ define $u_p(min)=min\{u_{pq};\ q\in B(p,x)\}$ and $w_q(min)= min\{w_{pq};\ p\in B(q,x)\}$.

Definition 2.1.3. *The feasible allocation $(u,w;x)$ is* ***P-non-discriminatory*** *if*

(f) $w_{pq}=w_q(min)$ *for all* $(p,q)\in C(x)$.

The feasible allocation $(u,w;x)$ is ***Q-non-discriminatory*** *if*

(g) $u_{pq}=u_p(min)$ *for all* $(p,q)\in C(x)$.

The feasible allocation $(u,w;x)$ is ***non-discriminatory*** *if the payoff functions u and w satisfy (f) and (g).*

Definition 2.1.4. *Let $S\subseteq P^*\cup Q^*$, $S\neq\phi$. The feasible labor time allocation x is* ***feasible for S*** *if, for every P-agent $p\in S$ and every Q-agent $q\in S$, $[B(p,x)-\{0\}]\subseteq S$ and $[B(q,x)-\{0\}]\subseteq S$. The feasible allocation $(u,w;x)$ is feasible for S if x is feasible for S.*

That is, under the assumptions above, x is feasible for S if no agent in this set interacts, at x, with real agents out of S.

For every $S\subseteq P^*\cup Q^*$ define $V(S)$ as being the set of feasible allocations that are feasible for S. That is,

(h) $V(S)=\{(u,w;x)\in\Sigma;\ x \text{ is feasible for } S\}$.

We assume that $V(\phi)=\phi$.

Observe that this function V can be identified with the characteristic function of the market. This function does not take into consideration the nature of the agreements, so it is the same for both markets.

For each $R\subseteq P^*$, $R\neq\phi$, and $T\subseteq Q^*$, $T\neq\phi$, the payoff $G(R\cup T)$ of coalition $R\cup T$ is given by

(i) $G(R\cup T)\equiv max\ \{\sum_{(p,q)\in RxT} a_{pq}\, x_{pq};\ x \text{ is feasible for } R\cup T\}$.

Define $G(S)=0$ if $S\subseteq P^*$ or $S\subseteq Q^*$. Also $G(\phi)=0$. That is, for all $S\subseteq P^*\cup Q^*$, $G(S)$ is the maximum income the players in S can get by themselves. According to this definition, *a feasible labor time allocation x is optimal if and only if* $G(P^*\cup Q^*)=\Sigma_{(p,q)\in PxQ}\, a_{pq}\, x_{pq}$.

REMARK 2.1.1. From Definition 2.1.4, if $(u,w;x)\in V(R\cup T)$, $R\subseteq P^*$, $R\neq\phi$, $T\subseteq Q^*$, $T\neq\phi$, then the players of $R\cup T$ achieve their total payoff and fill their quota of labor time without any interaction with real players out of $R\cup T$. The feasibility of $(u,w;x)$ then implies that $\sum_{p\in R,q\in T}U_p+W_q = \sum_{(p,q)\in RxT} a_{pq}\, x_{pq}$. By *(i)*, $\sum_{p\in R,q\in T} (U_p+W_q)\leq G(R\cup T)$. In particular, since any feasible allocation $(u,w;x)$ belongs to $V(P^*\cup Q^*)$, $\sum_{p\in P,q\in Q}(U_p+W_q)\leq G(P^*\cup Q^*)$, for all feasible allocations $(u,w;x)$. However, it is very easy to find an allocation that satisfies this expression and does not satisfy *(e)*, so it is not feasible. ■

Definition 2.1.5. *Let* E *be some non-empty set of feasible allocations. The feasible allocation* $\sigma=(\overline{u},\ \underline{w};x)\in E$ *is* ***P- optimal for set E*** *if* $\overline{U}_p\geq U'_p$ *for all* $p\in P$ *and all feasible allocations* $(u',w';x')$ *in* E. *Symmetrically, the feasible allocation* $\tau=(\underline{u},\ \overline{w};x)\in E$ *is* ***Q-optimal for set E if*** $\overline{W}_q\geq W'_q$ *for all* $q\in Q$ *and all feasible allocations* $(u',w';x')$ *in* E.

2.2. MATHEMATICAL MODEL: DEVIATION FUNCTION FORM

The assumption that the utilities are additively separable propitiates market rules, according to which, agents in a coalition can renegotiate among themselves while keeping the whole terms of some current agreements with current partners outside the group. These rules are specified by the feasible allocations. However, the renegotiations inside a partnership take into consideration the nature of the agreements. The rules which control such renegotiations are not specified by the feasible allocations. Thus the two markets provide the same sets of feasible allocations, but different rules. Therefore, the feasible allocations do not fully model the rules of the two markets. It turns out that the information on the nature of the agreements is crucial for the purpose of observing cooperative equilibrium allocations, as we can see in the following example.

Example 2.2.1. Consider $P=\{p_1\}$, $Q=\{q_1,\ q_2\}$, $r(p_1)=5=s(q_1)$, $s(q_2)=1$, $a_{11}=a_{12}=3$. Consider the allocation $\sigma=(u,w;x)$ where $x_{11}=5$, $x_{12}=0$, $x_{02}=1$; $u_{11}=1$, $w_{11}=2$, $w_{02}=0$. Then $U_1=5$, $W_1=10$ and $W_2=0$. This allocation is clearly feasible.

It is easy to verify that there is no way for p_1 to increase his total payoff by only trading with q_2. In order to increase his total payoff, p_1 must trade with both sellers. If the rules of the market allow that the number of negotiated units of labor time in the partnership $\{p_1,\ q_1\}$ could be reduced to 4, while keeping the division of the income a_{11}, p_1 and q_2 could reach the feasible allocation $\sigma'=(u',w';x')$, where $x'_{11}=4<5=x_{11}$,

x'_{12}=1, u'_{11}=1, u'_{12}=2, w'_{11}=2, w'_{12}=1. The total payoffs of p_1 and q_2 would be $U'_1=6>U_1$ and $W'_2=1>W_2$, respectively.

Nevertheless, if such kind of reformulation of current agreements is not allowed by the rules of the market, then the agreement between p_1 and q_1 should be nullified and a new agreement between these two agents should be proposed. It is easy to see that there are no prices that can increase the current total payoffs of the three agents: if q_1 receives more than 0 and q_2 receives more than 10 then p_1 will receive less than 5.

In sum, in a time-sharing assignment market operating in a cooperative environment, in which the agents can freely communicate to each other, if we do not know the kind of agreements that are allowed by the rules of the game, we cannot predict which allocations will or will not occur. The feasible allocation σ could be expected to occur in the rigid market but not in the flexible market. ■

The natural question is then: *How to model the rules of the game so that to capture the information on the nature of the agreements?*

As the example above illustrates, feasible allocation σ is a cooperative equilibrium allocation for the rigid market and it is not so for the flexible market. These conclusions cannot be obtained from the characteristic function *V,* since there is no way to capture the nature of the agreements from *V.* Thus, the two markets are indistinguishable under their representation in the characteristic function form. This makes inadequate the use of the characteristic function in modeling the rules of the rigid and the flexible markets for the purpose of observing cooperative equilibrium allocations.

The deficiencies inherent of the representation of these markets in the coalitional function form can be corrected with the use of the *deviation function form.* This is a mathematical model recently introduced in Sotomayor (2012) that can be used to represent a cooperative game of the sort we are treating here. The primitives of this model are the set of agents, the set of feasible allocations, and for each coalition $S\in P^*\cup Q^*$ and for each feasible allocation σ, the set $\phi_\sigma(S)$ of feasible allocations, called *set of feasible deviations from* σ *via* S. The feasible allocations express which decisions the players are allowed to take (rational decisions) and the *feasible deviations from* σ *via coalition* S reflect, in some sense, the actions that the players in S can take against σ and that are allowed by the rules of the game.

It is natural to require that $\phi_\sigma(\phi)=\phi$ and that $\phi_\sigma(S)\supseteq V(S)$ for all σ and all coalition S of real players. Also, if $\sigma' \in \phi_\sigma(S)$, then every player in S has at least one partner in $S\cup\{0\}$ under σ'; if a player in S has a real partner under σ', out of S, then this partner is also his partner at σ; if σ' and σ'' are feasible allocations which agree on the set of partners of the agents belonging to S, then if $\sigma' \in \phi_\sigma(S)$ then $\sigma'' \in \phi_\sigma(S)$ (internal consistency).

Thus, the players in S obtain a feasible deviation σ' from the feasible allocation σ by modifying σ through actions allowed by the rules of the game that take into account the nature of the agreements. Therefore, in the rigid market, the players in S can reach a *feasible deviation* σ' *from* σ by

(1) breaking some of their agreements at σ,

(2) keeping those ones at σ which were not broken and

(3) replacing the broken agreements at σ with a new set of agreements, which only involves agents in S.

Formally,

Definition 2.2.1. *Given a coalition* $S\subseteq P^*\cup Q^*$ *and a feasible allocation* $\sigma=(u,w;x)$, *the feasible allocation* $\sigma'=(u',w';x')$ *is a* ***feasible deviation from*** $\boldsymbol{\sigma}$ ***via*** $\boldsymbol{S}$ ***for the rigid market*** *if*

(j) *when* $[p\in S$ *and* $x'_{pq}>0]$ *then* $q\in S\cup\{0\}$ *or* $[x'_{pq}=x_{pq}$ *and* $u'_{pq}=u_{pq}]$*; when* $[q\in S$ *and* $x'_{pq}>0]$ *then* $p\in S\cup\{0\}$ *or* $[x'_{pq}=x_{pq}$ *and* $w'_{pq}=w_{pq}]$*;*

(k) *for every* $p\in S$, *there is some* $q\in B(p,x')$ *such that* $[x'_{pq}\neq x_{pq}$ *or* $u'_{pq}\neq u_{pq}]$; *for every* $q\in S$, *there is some* $p\in B(q,x')$ *such that* $[x'_{pq}\neq x_{pq}$ *or* $w'_{pq}\neq w_{pq}]$.

Condition *(j)* means that every player in S can keep some of his current agreements with partners out of S, and the new agreements are made with partners in $S\cup\{0\}$; condition *(k)* adds that every player in S makes, at least, one new agreement. If σ' is a feasible deviation from σ via S we say that S is a *deviating coalition*. We denote by $\phi_\sigma^R(S)$ the set of all feasible deviations from σ via S in the rigid market.

When agreements are flexible, a deviating coalition can do more than the rules specify when agreements are rigid. In fact, in *the flexible market*, there is one more action that the players in S can take to reach a *feasible deviation from* σ. They can

(2') reformulate the terms of their current agreements (which were not dissolved and were not maintained) with respect to the time allocation (by reducing the number of u.l.t and keeping the terms on the division of the income a_{pq}).

Formally,

Definition 2.2.2. *Given a coalition* $S \subseteq P^* \cup Q^*$ *and a feasible allocation* $\sigma=(u,w;x)$, *the feasible allocation* $\sigma'=(u',w';x')$ *is a* ***feasible deviation from*** σ ***via*** S ***for the flexible market*** *if*

(l) *when* $[p \in S$ *and* $x'_{pq}>0]$ *then* $q \in S \cup \{0\}$ *or* $[x_{pq} \geq x'_{pq}$ *and* $u'_{pq}=u_{pq}]$; *when* $[q \in S$ *and* $x'_{pq}>0]$ *then* $p \in S \cup \{0\}$ *or* $[x_{pq} \geq x'_{pq}$ *and* $w'_{pq}=w_{pq}]$;

(m) *for every* $p \in S$, *there is some* q *in* $B(p,x')$ *such that* $[x'_{pq}>x_{pq}$ *or* $u'_{pq} \neq u_{pq}]$; *for every* $q \in S$, *there is some* p *in* $B(p,x')$ *such that* $[x'_{pq}>x_{pq}$ *or* $w'_{pq} \neq w_{pq}]$.

We denote by $\phi_{\sigma}^{F}(S)$ the set of all feasible deviations from σ via S in the flexible market.

By Definitions 2.2.1 and 2.2.2, any feasible deviation from some $\sigma \in \Sigma$ via some coalition S of real players for the rigid market is also a feasible deviation from σ via S for the flexible market. Let ϕ^R (respectively, ϕ^F) be the set of all feasible deviations from feasible allocations via some coalition for the rigid market (respectively, flexible market). The deviation function form of the time-sharing assignment game with rigid agreements is then given by (P,Q,Σ,ϕ^R) and for the time-sharing assignment game with flexible agreements is given by (P,Q,Σ,ϕ^F).

2.3 COOPERATIVE EQUILIBRIUM: CONCEPT AND CHARACTERIZATION

Since Gale and Shapley (1962), the cooperative equilibrium allocations in matching markets are called *stable allocations*. The general idea was proposed in Sotomayor (2012) and extended to non-matching games. Roughly speaking, *a feasible allocation is stable if there is no coalition of players who can profitably and feasibly deviate from the given allocation by acting according to the rules of the game.*

This idea contrasts with that of core: *A feasible allocation is in the core if there is no coalition of players who can profitably and feasibly deviate from the given allocation by interacting only among themselves.*

For the purpose of observing cooperative equilibria, the appropriate model is that given by the deviating function form presented in section 2.2. Once this foundation is laid out, we are able to characterize the cooperative equilibrium allocations of both markets.

Translated to the rigid and flexible markets, we formally have:

Definition 2.3.1. *The feasible allocation* $\sigma=(u,w;x)$ *is* ***stable for the rigid*** *(respectively,* ***flexible****)* ***market*** *if there is no coalition* $S=R\cup T\neq\phi$, *with* $R\subseteq P^*$ *and* $T\subseteq Q^*$, *and a feasible allocation* $\sigma'=(u',w';x')$ *such that*

(i₁) $U'_p>U_p$ $\forall p\in R$ *and* $W'_q>W_q$ $\forall q\in T$ *and*

(i₂) $\sigma'\in\phi_{\sigma}^{R}(S)$ *(respectively,* $\phi_{\sigma}^{F}(S)$ *).*

If σ *is not stable it is called unstable.*

The standard notion of domination relation is the following:

Definition 2.3.2. *The feasible allocation* $\sigma'=(u',w';x')$ ***dominates*** *the feasible allocation* $\sigma=(u,w;x)$ *via coalition* $S=R\cup T\neq\phi$, *with* $R\subseteq P^*$ *and* $T\subseteq Q^*$, *if*

(i₁) $U'_p>U_p$ $\forall p\in R$ *and* $W'_q>W_q$ $\forall q\in T$ *and*

(i₂) $\sigma'\in V(S)$.

That is, the feasible allocation σ' dominates the feasible allocation σ via coalition S if every player in S prefers σ' to σ and the players of coalition S reach allocation σ' by

1. breaking all their current agreements, and
2. replacing their current agreements with a new set of agreements, which only involves players in S.

Then the players in S can profitably deviate from allocation σ and obtain σ' by interacting only among themselves. This is how a core allocation is defined. That is,

Definition 2.3.3. *A feasible allocation is in the* ***core*** *if it is not dominated by any other feasible allocation via some coalition. Such a coalition is called blocking coalition.*

The following two propositions will be useful. Proposition 2.3.1 gives a sufficient condition for a feasible allocation to be in the core. Proposition 2.3.2 asserts that every core allocation is individually rational. Consequently, every stable allocation for the rigid (respectively, flexible) market is individually rational.

Proposition 2.3.1. *Let (u,w;x) be a feasible allocation such that*

(n) $\sum_{p\in R} U_p + \sum_{q\in T} W_q \geq G(R\cup T)$, *for every* $R\subseteq P^*$ *and* $T\subseteq Q^*$.

Then, (u,w;x) is in the core.

Proposition 2.3.2. *If (u,w;x) is in the core then*

(o) $U_p \geq 0$ for all $p\in P$ *and* $W_q\geq 0$ for all $q\in Q$.

In order to get a characterization of the stable allocations in the rigid and in the flexible markets, we define two special cooperative solution concepts: setwise-stability and strong-stability. These notions are defined via two kinds of domination relations.

Definition 2.3.4. *The feasible allocation* $\sigma'=(u',w';x')$ ***quasi-dominates*** *the feasible allocation* $\sigma=(u,w;x)$ *via coalition* $S=R\cup T\neq\phi$, *with* $R\subseteq P^*$ *and* $T\subseteq Q^*$, *if*

(i₁) $U'_p > U_p$ $\forall p\in R$, $W'_q > W_q$ $\forall q\in T$ *and*

(i₂) if $p\in R$ *and* $x'_{pq}>0$ *then* $q\in T\cup\{0\}$ *or* $[x'_{pq}=x_{pq}$ *and* $u'_{pq}=u_{pq}]$; *if* $q\in T$ *and* $x'_{pq}>0$ *then* $p\in R\cup\{0\}$ *or* $[x'_{pq}=x_{pq}$ *and* $w'_{pq}=w_{pq}]$.

Definition 2.3.5. *A feasible allocation is* ***setwise-stable*** *if it is not quasi-dominated by any other feasible allocation via some coalition.*

Condition (i_1) of Definition 2.3.4 implies *(k).* Thus, if a feasible allocation $\sigma=(u,w;x)$ is quasi-dominated by a feasible allocation $\sigma'=(u',w';x')$ via some coalition S then *(j)* and *(k)* are satisfied, so Definition 2.2.1 implies that $\sigma'\in\phi_\sigma^R(S)$. By Definition 2.3.1, σ is unstable for the rigid market. Conversely, if σ is unstable for the rigid market then conditions (i_1) and (i_2) of Definition 2.3.1 imply conditions (i_1) and (i_2) of Definition 2.3.4, so σ is not setwise-stable by Definition

2.3.5. Therefore, *the stable allocations for the rigid market are the setwise-stable allocations.*

Definition 2.3.6. *The feasible allocation* $\sigma'=(u',w';x')$ ***strongly quasi-dominates*** *the feasible allocation* $\sigma=(u,w;x)$ *via coalition* $S=R\cup T\neq\phi$, *with* $R\subseteq P^*$ *and* $T\subseteq Q^*$, *if*

(i_1) $U'_p > U_p\ \ \forall p\in R,\ \ W'_q>W_q\ \ \forall q\in T$ *and*

(i_2) when $[p\in R$ *and* $x'_{pq}>0]$ *then* $q\in T\cup\{0\}$ *or* $[x_{pq}\geq x'_{pq}$ *and* $u'_{pq}=u_{pq}]$; *when* $[q\in T$ *and* $x'_{pq}>0]$ *then* $p\in R\cup\{0\}$ *or* $[x_{pq}\geq x'_{pq}$ *and* $w'_{pq}=w_{pq}]$.

Definition 2.3.7. *A feasible allocation is* ***strongly-stable*** *if it is not strongly quasi-dominated by any other feasible allocation via some coalition.*

By Definition 2.3.7, if a feasible allocation $\sigma=(u,w;x)$ is not strongly-stable then it is strongly-quasi-dominated by a feasible allocation $\sigma'=(u',w';x')$ via some coalition S. Then, Definition 2.2.2 implies that $\sigma'\in\phi^F_\sigma(S)$, and so, by Definition 2.3.1, σ is unstable for the flexible market. Conversely, if σ is unstable for the flexible market then conditions *(i_1)* and *(i_2)* of Definition 2.3.1 imply conditions *(i_1)* and *(i_2)* of Definition 2.3.6, so σ is not strongly-stable by Definition 2.3.7. Therefore, *the stable allocations for the flexible market are the strongly-stable allocations.*

Thus, we have proved the following proposition.

Proposition 2.3.3. (*i_1) A feasible allocation is stable for the rigid market if and only if it is setwise-stable; (i_2) a feasible allocation is stable for the flexible market if and only if it is strongly-stable.*

It is immediate from Definitions 2.3.2, 2.3.4 and 2.3.6 that domination implies quasi-domination, which implies strong quasi-domination. Thus, the core contains the set of setwise-stable allocations, which contains the set of strongly stable allocations. Equivalently, the core contains the set of stable allocations for the rigid market, which contains the set of stable allocations for the flexible market. Indeed, all these inclusions may be strict. In Example 2.2.1, allocation $\sigma=(u,w;x)$ is setwise-stable and is not strongly stable. Example 2.3.1 below illustrates a situation in which some core-allocation is not setwise-stable. Thus, the core may be bigger than the set of setwise-stable allocations.

Example 2.3.1. Consider $P=\{p_1,p_2\}$, $Q=\{q_1, q_2\}$ $r(p_1)=r(p_2)=s(q_2)=2$ and $s(q_1)=1$. The numbers a_{pq}'s are given by: $a_{11}=3$, $a_{21}=5$, $a_{12}=2$, $a_{22}=3$. The nature of the agreements is arbitrary. Consider the allocation $(u,w;x)$ where $x_{11}=0$, $x_{12}=1$, $x_{10}=1$, $x_{21}=x_{22}=1$, $x_{20}=0$ and $u_{12}=1$, $u_{10}=0$, $u_{22}=1$, $u_{21}=3$; $w_{12}=1$, $w_{21}=2$, $w_{22}=2$. The corresponding total payoffs are $U_1=1$, $U_2=4$, $W_1=2$ and $W_2=3$.

The values of the coalitions are given by: $G(p_1,q_1)=3$, $G(p_1,q_2)=4$, $G(p_2,q_1)=5$, $G(p_2,q_2)=6$, $G(p_1,q_1,q_2)=5$, $G(p_2,q_1,q_2)=8$, $G(p_1,p_2,q_1)=5$, $G(p_1,p_2,q_2)=6$, $G(p_1,p_2,q_1,q_2)=10$, $G(S)=0$ if $S\subseteq P$, or $S\subseteq Q$. It is a matter of verification that *(n)* is satisfied. Proposition 2.3.1 then implies that $(u,w;x)$ is in the core. However, $(u,w;x)$ is not setwise-stable. In fact, players p_1 and q_1 can increase their total payoffs if p_1 keeps his agreement with q_2, q_1 breaks his agreement with p_2, p_1 and q_1 work together 1 u.l.t. and receive for this labor, respectively, 0.5 and 2.5. ■

Examples 2.2.1 and 2.3.1 also illustrate that the interactions allowed among the members of a coalition, for the purpose of blocking an allocation, are not affected by the nature of the agreements. That is, both models have the same core. Of course, every blocking coalition is a deviating coalition for both, the rigid market and the flexible market, although the converse is not always true. As observed before, Example 2.2.1. illustrates that an allocation may be unstable under flexible agreements, stable under rigid agreements and so in the core of both markets.

REMARK 2.3.1. A third model for the time-sharing assignment game can be obtained by assuming that agents negotiate in block with their whole set of partners and disregard the individual payoffs they could obtain in each individual transaction. Under these rules, an outcome $(U,W;x)$ would be a vector of total payoffs, one total payoff for each player, plus a labor time allocation. Within this context, the outcome $(U,W;x)$ is feasible if $\sum_{p\in P^*,\, q\in Q^*}(U_p+W_q)\leq G(P^*\cup Q^*)$. This model is studied in Sotomayor (2002). We will refer to it as the **time-sharing assignment game with one-dimensional payoffs.**

It is easy see that the concept of core is the cooperative equilibrium concept for the time-sharing assignment game with one-dimensional payoffs. In fact, as it is shown in Sotomayor (2002), Definition 2.3.3 is equivalent to require that $\sum_{p\in R} U_p+\sum_{q\in T} W_q\geq G(R\cup S)$, for every $R\subseteq P^*$ and $T\subseteq Q^*$, and $\sum_{p\in P,\, q\in Q}(U_p+W_q)=G(P^*\cup Q^*)$. Therefore, the characteristic function V captures all details of the rules of the game that are relevant to the model. Then, $V(S)$ equals the set of the feasible deviations from σ via S, for all $\sigma \in \Sigma$ and all $S \subseteq P^* \cup Q^*$. Hence, the core concept is equivalent to the cooperative equilibrium concept for that model. ■

3. COMPETITIVE STRUCTURE OF THE TIME-SHARING ASSIGNMENT GAME

3.1 THE COMPETITIVE MARKET

The cooperative market corresponds to situations in which an individual or group of individuals is working cooperatively toward the achievement of some well-defined goal. In the competitive market, an individual or group of individuals is not only working toward different goals but are actually competing with each other. In this section we will analyze the competitive structure of the rigid and flexible markets.

We will be assuming that all u.l.t. are supplied by the sellers. Therefore, to be well defined, the competitive market should specify the set of goods, the set of agents and the demand correspondence of each buyer. Every *seller* wants to sell his units of labor to the *buyers* and all his units of labor have the same price (the *sellers* do not discriminate the *buyers*). In this context, the prices of the goods are not negotiated, but taken as given by the buyers who, according to their demand correspondences, demand a set of bundles of units of services which respects their quotas. The natural economic question is then to determine how should the goods be allocated to the buyers.

The natural solution concept is called *competitive equilibrium allocation*, which, informally, is a feasible allocation under which the bundle of goods allocated to a buyer belongs to him demand set at the given prices and all units of labor with a positive price are sold.

We will illustrate these notions by using a simpler competitive market which is obtained when the goods are indivisible. In this case, every seller q supplies $s(q)$ identical objects. Denote by Q^0 the set of all objects in the Economy (including the null objects, which are the objects owned by the dummy seller). The prices of all objects in Q^0 are announced, so that the objects supplied by a given seller have the same price. The rules of the competitive market should specify how the demand set of a buyer at a given price vector is defined. We will assume that a buyer p will demand the bundles of the $r(p)$ most preferred objects in Q^0 at prices p. These are the sets of $r(p)$ objects that maximize p´s total surplus among all subsets of Q^0 with $r(p)$ objects, assuming this total surplus is non-negative. Evidently, the objects of the demanded bundles by a buyer may produce distinct individual surpluses. The presence of the null objects in the Economy causes the demand set of a buyer to be non-empty, because he always has the option of demanding the null object so many times as needed to complete his quota.

We extend this notion to include the case where the goods are divisible. To do that, let a feasible assignment vector for p (or assignment vector for p, for short) be a vector of non-negative numbers $x_p \equiv (x_{pq})_{q \in Q}$ which satisfies *(a)* and such that $x_{pq} \leq s(q)$ for all $q \in Q$. The set of all feasible assignment vectors for p will be denoted by X_p. Clearly, if x is a feasible labor time allocation then x_p is a feasible assignment vector for p, for all $p \in P$.

A vector $\pi \in R^n_+$ is called feasible price vector or price vector, for short. That is, a price vector is a vector π of non-negative numbers, one coordinate for each seller, where π_q is the price of each u.l.t. offered by seller q.

In the competitive market, buyers have preferences over feasible assignment vectors. Given a price vector π, the preferences of agent p over feasible assignment vectors are completely described by the numbers a_{pq}'s. For any two assignment vectors for p, x_p and x'_p, p prefers x_p to x'_p at prices π if $\sum_{q \in Q} (a_{pq}-\pi_q)x_{pq} > \sum_{q \in Q} (a_{pq}-\pi_q)x'_{pq}$. Agent p is indifferent between these two assignment vectors if $\sum_{q \in Q} (a_{pq}-\pi_q)x_{pq} = \sum_{q \in Q} (a_{pq}-\pi_q)x'_{pq}$. The units of labor time supplied by q are acceptable to p at prices π if $a_{pq}-\pi_q \geq 0$.

Under the structure of preferences we are assuming, given a price vector π, each buyer p is able to determine which assignment vectors he would most prefer. The set of such assignment vectors is called *demand set of p at prices π* and denoted by $D_p(\pi)$. That is,

$$\mathbf{D_p(\pi)=\{x_p \in X_p;\ \sum_{q \in Q}(a_{pq}-\pi_q)x_{pq} \geq \sum_{q \in Q}(a_{pq}-\pi_q)x'_{pq}\ \ \forall\ x'_p \in X_p\}.}$$

Note that $D_p(\pi)$ is never empty, because p has always the option of buying the assignment vector x_p, with $x_{pq}=0$ for all $q \neq 0$ and $x_{p0}=r(p)$. Note also that, if $x_p \in D_p(\pi)$ and $x_{pq}>0$ then the units of labor time offered by q are acceptable to p.

Another way to defining the demand set of a buyer is to consider that a buyer demands all vectors of u.l.t. that can be feasibly assigned to him and that maximize all his individual surpluses. This approach will be discussed in section 6.

REMARK 3.1.1. If $x_p \in D_p(\pi)$ then $a_{pq}-\pi_q \geq a_{pt}-\pi_t$ for all sellers q and t such that $x_{pq}>0$ and $x_{pt}=0$. In fact, define the feasible assignment vector x'_p, where $x'_{pq^*}=x_{pq^*}$ for all $q^* \notin \{q,t\}$, $x'_{pq}=x_{pq}-\lambda \geq 0$, $x'_{pt}=\lambda$, where $\lambda>0$. Now use the definition of $D_p(\pi)$. ■

3.2. COMPETITIVE EQUILIBRIUM

The natural solution concept for the competitive market is that of *competitive equilibrium.*

Definition 3.2.1. *The pair (π,x) is a* ***competitive equilibrium*** *if (i_1) π is a price vector, (i_2) x is a feasible labor time allocation such that x_p is in the demand set of p at prices π, for all $p \in P$ and (i_3) $\pi_q=0$ if $x_{0q}>0$.*

If (π,x) is a competitive equilibrium then π is called ***a competitive equilibrium price vector*** *(or equilibrium price for short)* and we say that π is *compatible with* x or x *is compatible with* π. Labor allocation x is called ***competitive*** whenever it is compatible with a competitive equilibrium price. The corresponding money allocation for the Q-agents, that will also be denoted by π, is defined by $\pi_{pq}=\pi_q$ for all $(p,q) \in C(x)$. The corresponding money allocation for the P-agents is defined feasibly. The resulting feasible allocation $(u,\pi;x)$ is called a ***competitive equilibrium allocation*** and (u, π) is called a competitive equilibrium payoff.

It follows from Definition 2.1.3 that a competitive equilibrium allocation is P-non-discriminatory. Clearly, by the symmetry of the model, if we reverse the roles between buyers and sellers, we obtain that a competitive equilibrium allocation is Q-non-discriminatory.

4. THE EXISTENCE THEOREM

This section addresses the existence problem of the cooperative equilibrium allocations for the rigid and flexible markets. We need some preliminaries. Consider the primal linear programming problem (P1) of finding a matrix $x=(x_{pq})$ which maximizes

(A1) $\sum_{(p,q) \in PxQ} a_{pq}x_{pq}$

subject to:

(A2) $\sum_{q \in Q^*} x_{pq} \leq r(p)$ for all $p \in P^*$;

(A3) $\sum_{p \in P^*} x_{pq} \leq s(q)$ for all $q \in Q^*$;

(A4) $x_{pq} \geq 0$ for all $(p,q) \in P^* \times Q^*$,

The dual problem (P1)* is to find an m-vector $y=(y_p)_{p \in P^*}$ and an n-vector $z=(z_q)_{q \in Q^*}$ which minimizes

(B1) $\sum_{p \in P^*} r(p) y_p + \sum_{q \in Q^*} s(q) z_q$

subject to:

(B2) $y_p + z_q \geq a_{pq}$, for all $(p,q) \in P^* \times Q^*$;

(B3) $y_p \geq 0$, $z_q \geq 0$, for all $(p,q) \in P^* \times Q^*$.

Because we know that (P1) has a solution, we know that (P*1) must have an optimal solution[9]. By the Duality Theorem, for every solution x of (P1) and (y,z) of (P1*) we have that

$$\sum_{p \in P^*} r(p) y_p + \sum_{q \in Q^*} s(q) z_q = \sum_{PxQ} a_{pq} x_{pq} = G(P \cup Q).$$

Now, let x^* be an optimal solution for the linear programming problem (P1) and let (y,z) be an optimal dual solution. Then, by the Linear Programming Equilibrium Theorem or by the Complementary Slackness (see Gale, 1960), we can conclude that

(A) if $\sum_{q \in Q^*} x^*_{pq} < r(p)$ then $y_p=0$;

(B) if $\sum_{p \in P^*} x^*_{pq} < s(q)$ then $z_q=0$;

(C) if $x^*_{pq}=0$ then $y_p + z_q \geq a_{pq;}$

(D) if $x^*_{pq}>0$ then $y_p + z_q = a_{pq.}$

Let x be a labor time allocation obtained from x^* as follows: $x_{pq}=x^*_{pq}$ if $p \in P^*$ and $q \in Q^*$; if $\sum_{q \in Q^*} x^*_{pq} = k < r(p)$ for some $p \in P^*$ (respectively, $\sum_{p \in P^*} x^*_{pq} = k < s(q)$ for some $q \in Q^*$) then set $x_{p0}=r(p)-k$ (respectively, $x_{0q}=s(q)-k$). Clearly, x is an optimal labor time allocation. Also, given any optimal labor time allocation we can derive an optimal solution for (P1).

For all $p \in P^*$ and $q \in Q^*$ define $u_{pq}=y_p$ and $w_{pq}=z_q$ if $x_{pq}>0$; $u_{0q}= w_{0q}=0$ if $x_{0q}>0$ and $u_{p0}=w_{q0}=0$ if $x_{p0}>0$. Then, by (A) and (B) $u_p(min)=y_p$ and $w_q(min)=z_q$ for all $p \in P^*$ and $q \in Q^*$. The resulting allocation $(u,w;x)$ will be called *dual allocation* and (u,w) will be called *dual money allocation.* Then, **dual allocations always exist.** Furthermore, by construction, **any dual money allocation is compatible with any optimal labor time allocation.**

[9] Thompson (1980) considers a model in which the core is defined as the set of dual solutions of P1.

The existence proof of the stable allocations provides a new insight, not used yet in the continuous matching models: **competitive equilibrium allocations always exist and are stable for both models.** The stability of the competitive equilibrium allocations is the central result of this section (Theorem 4.6) and it is obtained via the characterization of these allocations as **the strongly-stable allocations which do not discriminate the buyers** (Theorem 4.5).

We need the following definitions.

Definition 4.1. *The pair (p,q) is* ***unsaturated*** *with respect to the labor time allocation x (unsaturated, for short) if $x_{pq}<r(p)$ and $x_{pq}<s(q)$.*

That is, (p,q) is unsaturated with respect to the labor time allocation x if no player in $\{p,q\}$ contributes all his labor time to the partnership. In particular, if $x_{pq}=0$ then $\{p,q\}$ is unsaturated.

NOTATION: Let $(u,w;x)$ be a feasible allocation. For every unsaturated pair (p,q), define $u_{p(q)}(min)\equiv min\{u_{pr};\ r\in B(p,x)-\{q\}\}$ and $w_{(p)q}(min)\equiv min\{w_{tq};\ t\in B(q,x)-\{p\}\}$. Thus, if $x_{pq}=0$, $u_{p(q)}(min)=u_p(min)$ and $w_{(p)q}(min)= w_q(min)$.

Definition 4.2. *The feasible allocation $(u,w;x)$ is* ***pairwise-strongly-stable*** *if it is feasible and*

(p) $u_{p(q)}(min) + w_{(p)q}(min) \geq a_{pq}$ *for all unsaturated pair $(p,q)\in P\text{x}Q$.*

Theorem 4.5 involves two non-obvious characterizations of the pairwise-strongly-stable allocations given by Lemmas 4.1 and 4.2, which are tied together in Theorem 4.3.

For a better understanding of these characterizations, consider a feasible allocation $(u,w;x)$ and a labor time allocation x'. We can construct a feasible allocation $(u',w';x')$ so that each agent q maintains his individual payoffs in the partnerships where he decreases or keeps his labor time contribution; if q increases his labor time contribution in (p,q) then, he obtains, for each unit of additional labor time, the minimum individual payoff among all individual payoffs he obtains with partners other than p. Call F the set of such feasible allocations derived from $(u,w;x)$. Of course, $(u,w;x)$ is in F. Also, if p is distinct from p', the feasible allocation in F that maximizes p's total payoff may be different from the feasible allocation in F that maximizes the total payoff of p'. However, Theorem 4.3 asserts that this is not the case if $(u,w;x)$ is strongly-stable. Moreover, $(u,w;x)$ is strongly-stable if and only if, for all $p\in P$, $U_p=\sum_{q\in B(p,x)}u_{pq}x_{pq}$, is the highest p's total payoff that can be generated by some feasible allocation in F.

Theorem 4.3 (Characterization of the strongly-stable allocations). *Let $(u,w;x)$ be a feasible allocation. The following assertions are equivalent*

(i_1) $(u,w;x)$ is strongly-stable;

(i_2) $(u,w;x)$ is pairwise-strongly-stable;

(i_3) for all $p\in P$ and feasible labor allocation x' we have that

(*) $U_p \geq \sum_{q\in B(p,x')}(a_{pq}-w'_{pq})x'_{pq}$,

where $w'_{pq}x'_{pq} = w_{pq}x'_{pq}$ *if* $x_{pq}\geq x'_{pq}$, $w'_{pq}x'_{pq} = w_{pq}x_{pq} + w_{(p)q}(min)(x'_{pq}-x_{pq})$ *if* $0<x_{pq}<x'_{pq}$ *and* $w'_{pq}x'_{pq}= w_{(p)q}(min)x'_{pq}$ *if* $0=x_{pq}<x'_{pq}$.

It is then immediate that:

Corollary 4.4. *Let $(u,w;x)$ be an allocation that is feasible and P-non-discriminatory. Then $(u,w;x)$ is strongly-stable if and only if, for all $p\in P$ and feasible labor allocation x', we have*

(**) $U_p \geq \sum_{q\in B(p,x')}(a_{pq}-w_{pq})x'_{pq}$.

Corollary 4.4 implies that, under a strongly-stable allocation that is P-non-discriminatory, every buyer is maximizing his total payoff by taking as given the prices of the u.l.t. supplied by the sellers. This is precisely how the concept of competitive equilibrium allocation is defined. Then, we have proved that the competitive equilibrium allocations are the strongly-stable allocations such that no Q-agent discriminates any P-agent. Formally,

Theorem 4.5 (Characterization of the competitive equilibrium allocations). *Let $(u,w;x)$ be a feasible allocation. Then $(u,w;x)$ is a competitive equilibrium allocation if and only if it is strongly-stable and* $w_{pq}=w_q(min)$ for all $(p,q)\in P\times Q$.

The characterization given by Theorems 4.5 does not take into account the nature of the agreements. More specifically, the nature of the agreements, which generates distinct cooperative structures in the time-sharing assignment game with multi-dimensional payoffs, does not have any effect on the competitive structure treated here.

By Theorem 4.5 and from the fact that every strongly-stable allocation is setwise-stable, the cooperative structure of the flexible market creates a bridge between the competitive and the cooperative structures of the time-sharing assignment game with rigid agreements: the competitive equilibrium allocations are also stable allocations under rigid agreements and they are in the core. Thus, we have proved Theorem 4.6 below.

Theorem 4.6 (Stability of the competitive-equilibrium allocations). *The competitive equilibrium allocations are stable in the rigid and in the flexible markets (and so they are in the core).*

However, the correlation between the cooperative and competitive structures is not the same in both markets. Example 4.1 illustrates that in the rigid market, the stable allocations which do not discriminate the buyers are not necessarily competitive. Thus, the fraction of the stable allocations that are competitive turns out to be smaller under rigid agreements than under flexible agreements.

Example 4.1. (Example 2.2.1 continued) Consider $P=\{p_1\}$, $Q=\{q_1, q_2\}$, $r(p_1)=5=s(q_1)$, $s(q_2)=1$, $a_{11}=a_{12}=3$. The allocation $(u,w;x)$ where $x_{11}=5$, $x_{12}=0$, $x_{02}=1$; $u_{11}=1$, $w_{11}=2$, $w_{02}=0$ is clearly P-non-discriminatory. As it was viewed in Example 2.2.1, $(u,w;x)$ is setwise-stable and it is not strongly-stable. Then, by Theorem 4.5, $(u,w;x)$ is not a competitive equilibrium allocation. ■

This kind of correlation between the competitive equilibrium allocations and the stable allocations, in both markets, is also different from that kind found in the multiple-partners assignment game. In that model the competitive equilibrium allocations can be created by “shrinking” the set of cooperative equilibrium allocations through an isotone function, which maps every stable allocation $(u,w;x)$ to a competitive equilibrium allocation $(u',w';x)$ where $w'_{pq}=w_q(min)$ for all $(p,q)\in P\times Q$ and u' is feasibly defined. The set of competitive equilibrium allocations is characterized as being the set of fixed points of that function. Such characterization fails to hold in the flexible market, as we can see in the example below.

Example 4.2. Consider $P=\{p_1,p_2\}$, $Q=\{q_1\}$, $r(p_1)=5=s(q_1)$, $r(p_2)=1$, $a_{11}=3$, $a_{21}=4$. The allocation $(u,w;x)$, where $x_{11}=4$, $x_{21}=1$, $x_{10}=1$; $u_{11}=1$, $u_{10}=0$, $u_{21}=1$, $w_{11}=2$, $w_{21}=3$, is clearly strongly-stable. However, the allocation $(u',w';x)$, where $w'_{11}=w'_{21}=2=min\{2,3\}$, $u'_{11}=1$, $u'_{10}=0$, $u'_{21}=2$, is not competitive since p_1 demands the whole amount of u.l.t. supplied by the seller. (Indeed this allocation is not in the core, since it is blocked by $\{p_1, q_1\}$). ■

Lemma 4.7. *Let $(u,w;x)$ be a strongly-stable allocation. Then x is an optimal labor time allocation.*

Note that if x is an optimal labor time allocation and $(u,w;x')$ is a strongly stable allocation with $x'\neq x$, then x is not necessarily compatible with (u,w). This is because u and w are not indexed according to x.

Proposition 4.8. *The set of non-discriminatory strongly-stable allocations coincides with the set of dual allocations.*

Theorem 4.9 (Existence Theorem) *The set of competitive equilibrium allocations, the set of stable allocations for the flexible market, the set of stable allocations for the rigid market and the core are always non-empty.*

Since any dual allocation is compatible with any optimal labor time allocation, we get the following stronger result.

Theorem 4.10 (Strong Existence Theorem) *Let x be an optimal labor time allocation. The set of competitive equilibrium allocations compatible with x, the set of stable allocations compatible with x, for the flexible and for the rigid models, and the set of core allocations compatible with x are non-empty.*

5. ALGEBRAIC STRUCTURE OF THE SOLUTION SETS

In this section we go beyond simple considerations provided by the set inclusion relation, and we also examine the connection between the three solution sets – core, the set of cooperative equilibrium allocations and the set of competitive equilibrium allocations - by focusing on their algebraic structure.

Let $E(x)$ be a non-empty subset of feasible allocations $(u,w;x)$, compatible with the labor time allocation x. Given a feasible allocation in $E(x)$, we can treat the array of individual payoffs of each player as a vector in some Euclidean space. We can then define a binary relation $\geq_P$ on $E(x)$ as follows. If the feasible allocations $(u,w;x)$ and $(u',w';x)$ belong to $E(x)$, $(u,w;x) \geq_P (u',w';x)$ if $u_{pq} \geq u'_{pq}$ for all $(p,q) \in C(x)$. Clearly, $\geq_P$ is reflexive and transitive. If $(u,w;x) \geq_P (u',w';x)$ then the feasibility of the allocations implies that $w_{pq} \leq w'_{pq}$ for all $(p,q) \in C(x)$, so the anti-symmetric property holds (if $(u,w;x) \geq_P (u',w';x)$ and $(u',w';x) \geq_P (u,w;x)$ then $(u,w;x)=(u',w';x)$). Therefore, $\geq_P$ defines a partial order relation in $E(x)$. (Observe that the total payoffs do not define a partial order relation. However, if $(u,w;x) \geq_P (u',w';x)$ then $U \geq U'$ and $W' \geq W$). Symmetrically we define $\geq_Q$. Now, let us define $(u^*,w_*;x)$ and $(u_*,w^*;x)$ as the money allocations in $E(x)$ such that:

$u^* \equiv u \vee u'$, $w_* \equiv w \wedge w'$; $u_* \equiv u \wedge u'$, $w^* \equiv w \vee w'$.

That is, for all $(p,q) \in C(x)$,

(u1) $u^*_{pq}=max\{u_{pq}, u'_{pq}\}$, $w_{*pq}=min\{w_{pq}, w'_{pq}\}$,

(u2) $u_{*pq}=min\{u_{pq}, u'_{pq}\}$ and $w^*_{pq}=max\{w_{pq}, w'_{pq}\}$.

Then, by denoting the meet and joint operations under $\geq_P$ by $\vee_P$ and $\wedge_P$, respectively, and under $\geq_Q$ by $\vee_Q$ and $\wedge_Q$, respectively, we can write:

(v1) $(u,w;x) \vee_P (u',w';x)= (u,w;x) \wedge_Q (u',w';x)=(u^*,w_*;x)$ and

(v2) $(u,w;x) \vee_Q (u',w';x)= (u,w;x) \wedge_P (u',w';x)= (u_*,w^*;x)$.

The set $E(x)$ is a *lattice* under any of two partial orders defined above if $(u^*,w_*;x)$ and $(u_*,w^*;x)$ are in $E(x)$ for every $(u,w;x)$ and $(u',w';x)$ in $E(x)$. That is, every two points in $E(x)$ have a supremum and an infimum in $E(x)$, according to the partial order relation that is being used. The lattice is *complete* if every subset of it has a supremum and an infimum. (See Birkhoff , 1973). Therefore, a compact lattice is a complete lattice.

Under these considerations, if $E(x)$ is a complete lattice then there exist one and only one supremum and one and only one infimum of $E(x)$. These two extreme points of $E(x)$ have an important meaning for the model. Even though the total payoffs of the players do not define the partial order relations $\geq_P$ and $\geq_Q$, the maximal feasible allocation under $\geq_P$ gives to each P-agent a payoff vector that is greater, in each component, than any other payoff vector that he can obtain under a feasible allocation in $E(x)$. Thus, this allocation gives to all P-agents as much total payoff as under any other allocation in $E(x)$, so it is the *P-optimal allocation for* $E(x)$. Symmetrically, the maximal allocation under $\geq_Q$ is the *Q-optimal allocation for* $E(x)$. (See Definition 2.1.5). Moreover, the optimal allocation for one of the sides of the market gives to the agents belonging to the other side the lowest total payoff that can be generated by any allocation in $E(x)$. That is, if $(\overline{u}, \underline{w};x)$ is P- optimal for $E(x)$ and $(\underline{u}, \overline{w};x)$ is Q-optimal for $E(x)$, then, for all $(p,q)\in PxQ$ and all $(u',w';x)\in E(x)$

(x1) $\overline{U}_p \geq U'_p$ and $\underline{W}_q \leq W'_q$,

(x2) $\underline{U}_p \leq U'_p$ and $\overline{W}_q \geq W'_q$

If the money allocations (u,w) and (u',w') correspond to different labor time allocations then, in general, it is no longer meaningful to consider, say $max(u,u')$, since u and u' are defined on different sets. Therefore, in this section, we investigate the algebraic structure of the set of core allocations, of the set of cooperative equilibrium allocations of each market and of the set of competitive equilibrium allocations, when these allocations are compatible with a given optimal labor time allocation. Theorem 4.10 implies that these sets are always non-empty.

Example 5.1 illustrates that the set of setwise-stable allocations and the set of core allocations, compatible with the same labor time allocation, are not always lattices. Consequently, the whole core and the whole set of setwise-stable allocations are not always lattices.

Example 5.1. Consider $P=\{p_1, p_2\}$, $Q=\{q_1, q_2\}$, $r(p_1)= r(p_2)=2$ and $s(q_1)=1$, $s(q_2)=3$; $a_{11}=4$, $a_{12}=1$, $a_{21}=4.5$ and $a_{22}=1.5$. Let the allocations $(u,w;x)$ and $(u',w';x)$ be given by $(x_{11}=x_{12}=1,\ x_{21}=0,\ x_{22}=2)$; $(u_{11}=1,\ u_{12}=1,\ u_{22}=1.5;\ w_{11}=3,\ w_{12}=0,\ w_{22}=0)$; $(u'_{11}=1.5,\ u'_{12}=0,\ u'_{22}=1;\ w'_{11}=2.5,\ w'_{12}=1,\ w'_{22}=0.5)$. It is a matter of verification that both allocations are in the core and are setwise-stable (use Proposition 2.3.1 to see that the allocations are in the core and then observe that if these allocations were quasi-dominated by some feasible allocation via some coalition, then this coalition would be a blocking coalition, which is a contradiction). The supremum of the two allocations under $\geq_P$, defined in *(u1)*, is given by $u^*_{11}=1.5$, $u^*_{12}=1$, $u^*_{22}=1.5$, $w_{*11}=2.5$, $w_{*12}=0$

and $w_{*22}=0$. This allocation is not in the core because it is blocked by $\{p_2,q_1,q_2\}$, so it is not setwise-stable. Hence, the set of core allocations and the set of setwise-stable allocations, compatible with x, are not lattices. ■

Continuing Example 5.1, we show that the feasible allocation $(u,w;x)$ is the P-optimal core allocation and the P-optimal setwise-stable allocation corresponding to x. Indeed, $(u,w;x)$ is the P-optimal core allocation and the P-optimal setwise-stable allocation. However, this allocation is not the supremum under $\geq_P$ for the set of core allocations and for the set of setwise-stable allocations when these allocations are compatible with x.

Example 5.1 (continued) Consider the market of example 5.1 again. It can be seen that the feasible allocation $(u,w;x)$ is the P-optimal setwise-stable allocation and the P-optimal core allocation associated to x. In fact, otherwise there would be some core allocation $(u'',w'';x)$ such that $U''_1>2$. (The total payoff of p_2 is already maximal, so $U''_2 \leq 3$.) Since the value of x is 8, then $U''_1+U''_2+W''_1+W''_2=8$, and so we would have

(1) $U''_2+W''_1+W''_2<6$.

On the other hand we must have

(2) $U''_2+W''_1\geq 4.5$ and $U''_2+W''_2\geq 3$,

if not (p_2,q_1) or (p_2,q_2) blocks the allocation. By (1) and (2) we get

(3) $W''_2<1.5$ and $W''_1<3$.

Then, by (1) and (3) it would be feasible that (p_2,q_1) and (p_2,q_2) be formed and 1 u.l.t. be allocated to each partnership. Therefore $\{p_2,q_1,q_2\}$ would block $(u'',w'';x)$, contradiction.

It is not hard to show that $(u,w;x)$ gives to the P-agents the highest total payoff among all core allocations. Then $(u,w;x)$ is a P-optimal core allocation and a P-optimal setwise-stable allocation. However, $(u,w;x)$ is not greater than or equal to $(u',w';x)$ under $\geq_P$, because $u_{11}<u'_{11}$, so the core and the set of setwise-stable allocations do not have a supremum under $\geq_P$. ■

Still using the market described in Example 5.1, we can see that the polarization of interests between the two sides of the market asserted in $(x1)$ is not always observed in the rigid market: The best setwise-stable allocation for the P-agents is not the worst setwise-stable allocation for the Q-agents. Indeed, there is no setwise-stable allocation that is the worst setwise-stable allocation for the Q-agents.

Example 5.1 (continued). Consider the market of Example 5.1 again. We have that $U_1>U'_1$ and $U_2>U'_2$ but $W_1>W'_1$ and $W_2<W'_2$. Therefore, $(u,w;x)$ is not the worst setwise-stable allocation for the Q-agents. We claim that there is no setwise-stable allocation that is the worst setwise-stable allocation for the Q-agents. In fact, otherwise there would be some setwise-stable money allocation (u'',w'') such that $W''_2=0$, and $W''_1<3$. However, $U''_2\leq 3$ due to the fact that $(u,w;x)$ is the P- optimal setwise-stable allocation. Therefore, (u'',w'') would be blocked by $\{p_2,q_1,q_2\}$ (find $\lambda>0$ so that $3-\lambda>W''_1$. Then give $3-\lambda$ to q_1 and $\lambda/2$ to q_2; the payoffs of p_2 are defined feasibly), so this money allocation would not be in the core and so it would not be setwise-stable. The same analysis shows that there is no core allocation that is the worst core allocation from the point of view of the Q-agents. ■

The sets of stable allocations of the rigid and flexible markets have distinct algebraic structures. Theorem 5.2 proves that the set of strongly-stable allocations, which are compatible with the same labor time allocation, is endowed with a complete lattice structure under both partial orders, $\geq_P$ and $\geq_Q$. Thus, in the flexible market, the set of stable allocations splits into several non-empty lattices, each one corresponding to an optimal labor time allocation. Theorem 5.3 implies that the set of competitive equilibrium allocations also has a non-standard algebraic structure of a union of complete lattices. Each of these lattices is a sub-lattice of the corresponding lattice of strongly-stable allocations.

Lemma 5.1. *Let $(u,w;x)$ and $(u',w';x)$ be strongly-stable allocations. Then $(u^*,v_*;x)$ and $(u_*,v^*;x)$, defined in (u1) and (u2), are strongly-stable allocations.*

Theorem 5.2 (Algebraic structure of the set o strongly-stable allocations) *Let x be an optimal labor time allocation. Then,*

a) the set of the strongly-stable allocations compatible with x is a complete lattice under both partial orders $\geq_P$ and $\geq_Q$;

b) this set has a P-optimal and a Q-optimal allocations and *c) properties (x1) and (x2) hold.*

NOTATION: Given the optimal labor time allocation *x,* denote by *A(x)* the set of competitive equilibrium allocations compatible with *x*.

Theorem 5.3 ((Algebraic structure of the set o competitive equilibrium allocations)
Let x be an optimal labor time allocation. Then,
a) the set A(x) is a complete lattice under both partial orders $\geq_P$ and $\geq_Q$;
b) there always exist the P-optimal and Q-optimal competitive equilibrium allocations for A(x) and
c) properties (x1) and (x2) hold.

Our next results prove that every two lattices of competitive equilibrium allocations are in one-to-one correspondence through a function which preserves the total payoff of the agents. The extreme points of the first lattice are mapped to the corresponding extreme points of the other lattice. Thus, the supremum of each lattice is a P-optimal competitive equilibrium allocation if we use the partial order $\geq_P$, and it is a Q-optimal competitive equilibrium allocation if we use the partial order $\geq_Q$.

Define the function $f_{x'}$: $A(x) \to A(x')$ by $f_{x'}(u,w;x)=(u',w;x')$ where *u'* is defined feasibly. Proposition 5.4 implies that $f_{x'}$ is well defined and preserves the total payoffs of the agents.

Proposition 5.4. *Let (u,w;x) be a competitive equilibrium allocation in A(x) and let x' be an optimal labor time allocation. Set* $f_{x'}(u,w;x)\equiv(u',w';x')$. *Then, (u',w';x') is a competitive equilibrium allocation in A(x'). Furthermore,* $U_p=U'_p$ *for all* $p\in P$ *and* $W_q=W'_q$ *for all* $q\in Q$. *Symmetric results hold if we reverse the roles between P and Q agents.*

REMARK 5.1 Proposition 5.4 plus Lemma 4.7 imply that there is a Cartesian product structure in the set of competitive equilibria: *(w.x)* is a competitive equilibrium if and only if *w* is an equilibrium price and *x* is an optimal labor time allocation. ■

The feasibility of the competitive equilibrium allocations implies that, if $(u,w;x)\neq(u',w';x)$ then $w\neq w'$, so $f_{x'}(u,w;x) \neq f_{x'}(u',w';x)$. Then, $f_{x'}$ is one-to-one.

Clearly, $f_{x'}$ is onto *A(x')*. Theorem 5.5 below asserts that $f_{x'}$ maps the extreme points of *A(x)* into the corresponding extreme points of *A(x')*.

Theorem 5.5. *Let (u,w;x) be the P-optimal(respectively, Q-optimal) competitive equilibrium allocation for A(x). Let x' be any optimal labor time allocation. Then,* $f_{x'}$*(u,w;x) is the P-optimal(respectively, Q-optimal) competitive equilibrium allocation for A(x').*

By Theorem 5.6, below, the *P*-optimal (respectively, *Q*-optimal) competitive equilibrium allocations of the lattices give to the players the same total payoff. Thus, the *P*-optimal (respectively, *Q*-optimal) competitive equilibrium allocation of any lattice is a *P*-optimal (respectively, *Q*-optimal) competitive equilibrium allocation of the whole set of competitive equilibrium allocations.

Theorem 5.6. *Let* $(\overline{u}, \underline{w}; x)$ *and* $(\overline{u}', \underline{w}'; x')$ *be the P-optimal competitive equilibrium allocations of A(x) and A(x'), respectively. Then,* $\overline{U} = \overline{U}'$ *and* $\underline{W} = \underline{W}'$.

Clearly, Theorem 5.6 also holds if the allocations are *Q*-optimal competitive equilibrium allocations or *P*-optimal (respectively, *Q*-optimal) strongly-stable and non-discriminatory for *Q*.

It follows immediately from Theorems 5.3 and 5.6 that

Corollary 5.7 *There always exist P-optimal and Q-optimal allocations for the whole set of competitive equilibrium allocations.*

The algebraic structure of the set of competitive equilibrium allocations is distinct from that of the strongly-stable allocations. As it is illustrated in the example below, the algebraic structure of the set of strongly stable allocations does not guarantee the existence of optimal stable allocations for each side of the flexible market.

Example 5.2 (*P*-optimal allocations of two lattices of strongly-stable allocations generating different total payoffs for the *P*-agents). Consider the market where $P=\{p_1,$

$p_2\}$, $Q=\{q_1, q_2\}$, $r(p_1)=3$, $r(p_2)=1$, $s(q_1)=2$ and $s(q_2)=2$. The matrix a is given by: $a_{11}=3$, $a_{12}=2$, $a_{21}=3$, $a_{22}=2$. There are two optimal labor time allocations x' and x'', where $x'_{11}=2, x'_{12}=1$, $x'_{21}=0$, $x'_{22}=1$ and $x''_{11}=1, x''_{12}=2$, $x''_{21}=1$, $x''_{22}=0$. Consider allocation $(u',w'; x')$ where $u'_{11}=2, u'_{12}=2$, $u'_{22}=2$, $w'_{11}=1$, $w'_{12}=0$, $w'_{22}=0$. It is a matter of verification that $(u',w';x')$ is strongly-stable. On the other hand, allocation $(u'',w'';x'')$, where $u''_{11}=3$, $u''_{12}=2$, $u''_{21}=2$, $w''_{11}=0$, $w''_{21}=1$, $w''_{12}=0$, is also strongly-stable. We claim that $(u',w';x')$ and $(u'',w'';x'')$ are P-optimal strongly-stable allocations associated to x' and x'', respectively. In fact, observe that for every strongly-stable allocation $(u,w;x')$ we must have that $u_{22}+w_{11}\geq a_{21}=3$ (this is because (p_2,q_1) is unsaturated with respect to x'). On the other hand $a_{22}=2$, so $u_{22}\leq 2$. Then, $w_{11}\geq 1$, so $u_{11}\leq 2$ by feasibility; also, $u_{12}+w_{12}=2$ implies $u_{12}\leq 2$. Hence $(u',w';x')$ is the P-optimal strongly-stable allocation associated to x'. For the other allocation observe that (p_1,q_1) is unsaturated with respect to x'', so for every strongly-stable allocation $(u,w;x'')$ we must have that $u_{12}+w_{21}\geq a_{11}=3$. But $u_{12}\leq 2$ by feasibility, then $w_{21}\geq 1$, so $u_{21}\leq 2$. Feasibility also implies that $u_{11}\leq 3$. Hence $(u'',w'';x'')$ is the P-optimal strongly-stable allocation associated to x''. These two allocations generate the following total payoffs for the P-agents: $U'_1=6$, $U'_2=2$; $U''_1=7$, $U''_2=2$. The total payoffs to p_1 are distinct.■

REMARK 5.2. By Remark 5.1, any equilibrium price vector is compatible with any labor time allocation. Thus, given a labor time allocation x, the set of competitive equilibrium prices is the projection, on the space of the individual payoffs of the sellers, of the lattice of competitive equilibrium allocations compatible with x, so it is a lattice under the partial order $\geq_Q$. Then, there is one and only one P-optimal (respectively, Q-optimal) competitive equilibrium price. Consequently, among the competitive equilibrium price vectors there is a unique one that is at least as small in every component as any other competitive equilibrium price vector. It is called *minimum competitive equilibrium price.* The *maximum competitive equilibrium price,* with symmetrical properties, also exists.■

REMARK 5.3. The preferences of the Q-agents also define the partial order relation $\geq^*_Q$ in the set of competitive equilibrium prices: $w\geq^*_Q w'$ if and only if $W_q \geq W'_q$ for all q. (observe that the anti-symmetric property fails to hold for this binary relation when it is used in the set of competitive equilibrium allocations, so it does not define a partial order on this set). Clearly, the set of competitive equilibrium prices is a lattice under $\geq^*_Q$. More generally, the set of dual allocations is a lattice under both partial order relations $\geq^*_P$ and $\geq^*_Q$ defined as follows: $(u,w;x)\geq^*_P(u',w';x')$ if and only if $U_p\geq U'_p$ for all $p\in P$ and $(u,w;x)\geq^*_Q(u',w';x')$ if and only if $W_q\geq W'_q$ for all $q\in Q$. ■

6. FINAL REMARKS AND RELATED WORK

The continuous two-sided matching markets which have been presented in the literature can be viewed as generalizations of the assignment game of Shapley and Shubik (1972). The main feature of these markets is that they are endowed with both cooperative game and competitive market game structures and can be fully represented in the characteristic function form. The intuitive idea of cooperative equilibrium for these models, called stability since Gale and Shapley (1962), was formalized in Sotomayor (2012). It has been characterized as a refinement of the core concept, called setwise-stability (see Sotomayor, 1999-b). In the models with one-dimensional payoffs, setwise-stability is equivalent to the core concept. In the multiple-partners game, due to the fact that the payoffs are multi-dimensional, setwise-stability is equivalent to the strong-stability concept, which is equivalent to the concept of pairwise-stability. The set of stable allocations is a proper subset of the core and it is always a non-empty complete lattice. Furthermore, the core is always non-empty, but it is not endowed, necessarily, with a lattice structure. The competitive approach supposes the market operating as an exchange economy. The concept of competitive equilibrium allocation is closely related to the traditional concept of equilibrium from standard microeconomic theory and it is an extension of the concept due to Gale (1960).[10] The set of competitive equilibrium allocations is a non-empty sub-lattice of the lattice of stable allocations and it is characterized as the set of stable allocations in which no seller discriminates any buyer.

In the present paper we introduced the continuous time sharing assignment game with multi-dimensional payoffs. The novelty is that this game is well defined for the purpose of observing core allocations or competitive equilibrium allocations, but it is incomplete if one wishes to observe cooperative equilibrium allocations. To be a complete model, the time-sharing assignment game with multi-dimensional payoffs requires that the rules specify the nature of the agreements inside each buyer-seller partnership. For our purposes it was simpler to work with two separate markets, the rigid market and the flexible market, defined according to the nature of the agreements. One peculiarity of these market games is that they have the same sets of feasible allocations, the same core and the same sets of competitive equilibrium allocations, which makes them indistinguishable under their representation in the characteristic function form. Nevertheless, they are endowed with different sets of cooperative

[10] The concept of competitive equilibrium allocation for many-to-many matching models was introduced in Sotomayor (2007).

equilibrium allocations. Therefore, unlike the previous matching models studied in the literature, these models cannot be fully represented in the characteristic function form. The adequate mathematical model for both markets is given by the **deviation function form** proposed in Sotomayor (2012).

Having defined the concepts of dominance, quasi-dominance and strong-quasi-dominance for the time-sharing assignment game, we identified three solution sets, the core, the set of setwise-stable allocations and the set of strongly-stable allocations, possibly distinct, each a super-set of the next. Surprisingly, the concept of setwise-stability, viewed as a general definition of stability since Sotomayor (1999), does not capture the idea of stability in the flexible market. This concept characterizes stability in the rigid market but not in the flexible market. The concept of stability for the flexible market is characterized by the concept of **strong-stability,** which is equivalent to the pairwise-strong-stability concept.

Adapted to the assignment game and to the multiple-partners assignment game, the concepts of setwise-stability and strong-stability coincide and characterize stability. With the changes in the rules of the multiple-partners assignment game, incorporated into the time-sharing assignment game, setwise-stability comes up as a new solution concept, different from the core concept, from the strong-stability concept and from the pairwise-stability concept. It plays the role of an intermediate solution concept: the set of setwise-stable allocations may be smaller than the core and may be bigger than the set of strongly-stable allocations.

From the technical point of view, the use of the competitive structure to prove the non-emptiness of the set of cooperative equilibrium allocations (Theorems 4.5 and 4.6) provides new insights that can be applied to other models. This methodology is unusual in matching models and cannot be applied to the discrete matching markets. An open problem in the literature of matchings is to know if the core of the discrete many-to-many matching model, with substitutable preferences, defined in Sotomayor (1999-b), is always non-empty. In that model it is not possible to define a related competitive market.

The proof that the strongly-stable allocations, which do not discriminate the P-agents, characterize the competitive equilibrium allocations when the buyers are the P-gents is not trivial and it is the most ingenious part of this paper. The fact that the set of non-discriminatory strongly-stable allocations can be identified with the set of the dual solutions of the transportation problem conveniently defined is also not obvious. The dual solutions always exist, which leads to the existence theorem. On the other hand, every dual allocation is compatible with any optimal labor time allocation. A stronger result was then obtained: *all three solution sets are not only non-empty, but the restriction of any of them to any optimal labor time allocation is also non-empty*.

From the conceptual point of view, the algebraic structure of the solution sets helps to better understand the correlation between the cooperative equilibrium allocations and the competitive equilibrium allocations, and lends new insights to the theory of two-sided matching markets. We showed that none of the solution sets is a lattice. The rules in each market generate distinct algebraic structures for the sets of cooperative allocations compatible with the same optimal labor time allocation. In fact, the lattice property only holds in the flexible market. However, we cannot guarantee the existence of the *P*-optimal and the *Q*-optimal stable allocations in either market. This sort of things is different in the set of competitive equilibrium allocations. This set is a union of non-disjoint sub-lattices of the corresponding lattices of strongly-stable allocations. Nevertheless, *the P-optimal and the Q-optimal allocations for the whole set of competitive equilibrium allocations always exist.*

From the practical point of view, the existence of the *P*-optimal and the *Q*-optimal competitive equilibrium allocations permits to treat situations in which we face the problem of choosing, for each market, some convenient stable allocation, which always exists and keeps its characteristics in every market. Consider, for example, the problem of investigating the effects on the agents' payoffs caused by the entrance of new agents in a given two-sided matching market. This is a problem of economic interest that has been treated by several authors. The comparison between an arbitrary core point of the original market and an arbitrary core point after the entrance of these agents may be meaningless. The central issue of such a study is the choice of a stable allocation x in the original market, and a stable allocation y in the new market, such that y captures the effects caused on x by the entrance of the new agents in the market.

The simplest way to select such allocations is to consider an allocation rule which applies before and after the entrance of the new agents. That is, these points should be such that after the entrance of the new agents in the market, the agents who were in the market can get allocation y by continuing to do the same sort of things they were doing to reach allocation x.

In the literature, meaningful comparative static results of adding agents to matching markets have always been obtained under the assumption that agents are allocated according to one of the extreme points of the lattice of the stable payoffs. This is because these allocations always exist and can be obtained by means of some well known algorithms, which can be reproduced after the entrance of the new agents in the market.

In both time-sharing assignment markets, the set of stable allocations is not always a complete lattice, so we cannot guarantee the existence of the optimal-stable allocations. However, this is not a hindrance to get meaningful comparative static results in both rigid and flexible markets. For example, if agents are allocated according to some *P*-optimal competitive equilibrium allocation, the effect caused on this allocation by the entrance of agents in the market can be viewed if we compare any of the *P*-optimal competitive equilibrium allocations of the original market with any of the *P*-optimal competitive equilibrium allocations of the new market. In both models, these allocations always exist, they are stable and they preserve their characteristic of being the *P*-optimal competitive equilibrium allocations.

Another application of our results is the design of an allocation mechanism which yields a stable allocation. The usual procedure in matching markets is to allocate the agents according to the *P*-optimal or *Q*-optimal stable allocations. Here again, the use of the optimal competitive equilibrium allocations makes the problem treatable.

It can be easily verified that all results of the present paper could be obtained if we required that the numbers $r(p)$'s, $s(q)$'s and x_{pq}'s were integers. The market of buyers and sellers of indivisible goods, in which the quota of a seller is the number of identical objects he owns, the quota of a buyer is the maximum number of objects he can acquire and a buyer is allowed to purchase more than one item from the same seller fits well in this model. Within this context, the multiple-partners assignment game can be viewed as the time-sharing assignment game where each buyer can acquire one item, at most, from a seller.

Some related work has been presented in the literature. Sotomayor (2010) has extended the time-sharing assignment game with rigid agreements to a non-matching coalitional game, in which players form coalitions of any size. The concept of stability was identified with the appropriate version of the setwise-stability concept given here. It was proved in that paper that the core may be bigger than the set of stable allocations.

It is well-known from Shapley and Shubik (1972) that the set of dual allocations for the assignment game coincides with the core and with the set of competitive equilibrium allocations. Inspired on this model, Thompson (1980) formulated a version of the time sharing assignment game with indivisible goods. This author believed that the equivalence between the dual allocations and the core observed in the assignment game persisted in his model and called "core" the set of dual allocations for his model. In Thompson's model, the set of dual allocations is not the core, as usually defined. In the case in which the agents' payoffs are one-dimensional, studied in Sotomayor (2002), or in the present case, where the agents' payoffs are multi-dimensional, the core contains the set of dual allocations and may be bigger than this set.

Shapley and Shubik also showed that the core and the set of competitive equilibrium allocations have the property of complete lattices and are given by the Cartesian product of the corresponding set of money allocations by the set of optimal matchings. The results presented in the previous sections make it evident that these two properties are not robust to the introduction of time into the assignment game. However, we showed in Appendix II, that it is possible to define another competitive market for the time-sharing assignment game (competitive market with non-discriminatory demand) whose set of competitive equilibrium allocations have the same characteristic properties as those of the assignment game: it coincides with the set of dual allocations, it has the lattice property and it is the Cartesian product of the set of competitive equilibrium money allocations by the set of optimal money allocations. This competitive market, when considered under the rules of the assignment game, coincides with the competitive market naturally defined for that model.

A variation of the market with buyers and sellers of indivisible goods described above was obtained in Jaume *et al* (2007) by allowing that the objects of a seller may be distinct. These authors concentrated their analysis on the algebraic structure of the set of competitive equilibrium price vectors, rather than on the algebraic structure of the set of competitive equilibrium allocations. Their competitive equilibrium concept is closely related to that, presented in Appendix II, for the market with non-discriminatory demand. They prove that this set preserves the lattice structure that is observed in the previous models.

Camiña (2006) studies the particular case in which a unique seller owns all objects, not necessarily identical, and each buyer wants to buy one object at most. This author shows that the set of core allocations is a non-empty complete lattice under the partial order defined by the preferences of the buyers and may be different from the set of competitive equilibrium allocations.

Finally, we would like to emphasize that the study developed here raises a relevant issue which comes out when a new matching model is proposed: What is stability? Which feasible allocations are the cooperative equilibrium allocations? The deviation function form reinforces the idea that, given a stable allocation, no coalition can come with a preferred alternative, which can be obtained by feasible deviations from the given allocation. Such representation allows identifying the correct characterization of the stable allocations for the rigid and flexible markets. However, our results establish that setwise-stability is not the general characterization of the concept of stability. This fact suggests that a general concept that captures the intuitive idea of cooperative equilibrium in any matching model may even not exist.

APPENDIX I: PROOFS

Proposition 2.3.1. *Let $(u,w;x)$ be a feasible allocation such that*

(n) $\sum_{p\in R} U_p + \sum_{q\in T} W_q \geq G(R\cup T)$, *for every* $R\subseteq P^*$ *and* $T\subseteq Q^*$.

Then, $(u,w;x)$ is in the core.

Proof. If the feasible allocation $(u,w;x)$ was dominated by some feasible allocation $(u',w';x')$ via some coalition $R\cup T$, Definition 2.3.2-(i_2) would imply that $(u',w';x')\in V(R\cup T)$. By Remark 2.1.1, $\sum_{p\in R} U'_p + \sum_{q\in T} W'_q \leq G(R\cup T)$. Definition 2.3.2-($i_1$) then would imply $\sum_{p\in R} U_p + \sum_{q\in T} W_q < G(R\cup T)$, which is a contradiction■.

Proposition 2.3.2. *If $(u,w;x)$ is in the core then*

(o) $U_p \geq 0$ for all $p\in P$ *and* $W_q \geq 0$ for all $q\in Q$.

Proof. This is immediate from the fact that if, say $U_p<0$ for some agent $p\in P$, then any feasible allocation that gives zero amount of money to p and allocates his quota of labor time to the dummy *Q*-agent, would dominate $(u,w;x)$ via coalition $S=\{p\}$. ■

The following lemmas are used to prove Theorem 4.3. Lemma 4.1 characterizes the pairwise-strongly-stable allocations as the feasible allocations where the total payoff of every buyer is a maximum among all feasible allocations in *F*. Then Lemma 4.2 characterizes the stable allocations of the flexible market as the pairwise-strongly-stable allocations.

Lemma 4.1. *Let $(u,w;x)$ be a feasible allocation. Then $(u,w;x)$ is pairwise-strongly-stable if and only if for all $p\in P$ and feasible labor allocation x' we have that*

(*) $U_p \geq \sum_{q\in B(p,x')}(a_{pq}-w'_{pq})x'_{pq}$,

where $w'_{pq}x'_{pq} = w_{pq}x'_{pq}$ *if* $x_{pq} \geq x'_{pq}$, $w'_{pq}x'_{pq} = w_{pq}x_{pq} + w_{(p)q}(min)(x'_{pq}-x_{pq})$ *if* $0<x_{pq}<x'_{pq}$ *and* $w'_{pq}x'_{pq}= w_{(p)q}(min)x'_{pq}$ *if* $x'_{pq}>0$ *and* $x_{pq}=0$.

Proof. Suppose $(u,w;x)$ is pairwise-strongly-stable but there is some feasible labor allocation x' and $p\in P$ such that (*) is not satisfied. Then,

(1) $U_p < \sum_{q\in B(p,x')}(a_{pq}-w'_{pq})x'_{pq}$.

Define $I=\{q\in Q;\ x_{pq} \geq x'_{pq}>0\}$, $J=\{q\in Q;\ x_{pq} < x'_{pq}\}$ and $K=\{q\in Q;\ x_{pq} >0, x'_{pq}=0\}$.

Then, $\sum_{q\in I} u_{pq}x_{pq} + \sum_{q\in J\cap B(p,x)} u_{pq}x_{pq} + \sum_{q\in K} u_{pq}x_{pq} < \sum_{q\in I} (a_{pq}-w_{pq})x'_{pq} + \sum_{q\in J}(a_{pq}-w'_{pq})x'_{pq} = \sum_{q\in I} u_{pq}x'_{pq} + \sum_{q\in J\cap B(p,x)} [(a_{pq}-w_{pq})x_{pq} + (a_{pq}- w_{(p)q}(min))(x'_{pq}-x_{pq})] + \sum_{q\in J-B(p,x)} (a_{pq}- w_{(p)q}(min))x'_{pq} =\sum_{q\in I} u_{pq}x'_{pq} + \sum_{q\in J\cap B(p,x)}u_{pq}x_{pq} +\sum_{q\in J} (a_{pq}- w_{(p)q}(min))x'_{pq} -\sum_{q\in J\cap B(p,x)}(a_{pq}- w_{(p)q}(min))x_{pq}= \sum_{q\in I} u_{pq}x'_{pq}+ \sum_{q\in J\cap B(p,x)}u_{pq}x_{pq}+\sum_{q\in J}(a_{pq}- w_{(p)q}(min))(x'_{pq}-x_{pq})$, so

(2) $\sum_{q\in I} u_{pq}(x_{pq} - x'_{pq}) + \sum_{q\in K} u_{pq}x_{pq} < \sum_{q\in J} (a_{pq}- w_{(p)q}(min))(x'_{pq}-x_{pq})$.

Now, let $u_{pr}=min\{u_{pq};q\in I\cup K\}$. Since $\sum_{q\in I}(x_{pq} - x'_{pq}) = (r(p) - \sum_{q\in J\cup K} x_{pq}) - (r(p)-\sum_{q\in J} x'_{pq})= \sum_{q\in J} (x'_{pq} - x_{pq}) - \sum_{q\in K} x_{pq}$ it follows that $u_{pr}\sum_{q\in J} (x'_{pq} - x_{pq}) = u_{pr} \sum_{q\in I}(x_{pq} - x'_{pq}) +u_{pr}\sum_{q\in K} x_{pq} \leq \sum_{q\in I} u_{pq}(x_{pq} - x'_{pq}) + \sum_{q\in K} u_{pq}x_{pq}$.

By (2) we have

$u_{pr}\sum_{q\in J} (x'_{pq} - x_{pq}) < \sum_{q\in J} (a_{pq}- w_{(p)q}(min))(x'_{pq}-x_{pq})$.

Then,

$\sum_{q\in J}(a_{pq}- w_{(p)q}(min) - u_{pr})(x'_{pq}-x_{pq})>0$, so we must have $(a_{pq}- w_{(p)q}(min) - u_{pr}) >0$ for some $q\in J$. But $r\neq q$ because $r\in I\cup K$ and $q\in J$. Then $u_{pr} \geq u_{p(q)}(min)$ and $0< a_{pq}- w_{(p)q}(min) - u_{pr} \leq a_{pq} - w_{(p)q}(min) - u_{p(q)}(min)$, so $u_{p(q)}(min) + w_{(p)q}(min)< a_{pq}$, which contradicts the assumption that $(u,w;x)$ is pairwise strongly-stable. Hence (*) is satisfied for all $p\in P$ and feasible labor allocation x'.

In the other direction, let (p,q^*) be an unsaturated pair (with respect to x). Then $x_{pq^*}<r(p)$ and $x_{pq^*}<s(q^*)$. Set $u_{pm} \equiv u_{p(q^*)}(min)$. Let λ be some positive number such that $x_{pm}- \lambda \geq 0$, $x_{pq^*} + \lambda\leq r(p)$ and $x_{pq^*} + \lambda\leq s(q^*)$. Consider a feasible

labor time allocation x' such that $x'_{pq^*} = x_{pq^*} + \lambda$, $x'_{pm} = x_{pm} - \lambda$, $x'_{pk} = x_{pk}$ for all $k \notin \{q^*, m\}$. Then we have

$\sum_{k\in B(p,x)-\{q^*,m\}} u_{pk}x_{pk} + u_{pm}x_{pm} + u_{pq^*}x_{pq^*} = \sum_{k\in B(p,x)} u_{pk}x_{pk} = U_p \geq \sum_{k\in B(p,x)-\{q^*,m\}} (a_{pk} - w_{pk})\, x_{pk} + (a_{pm} - w_{pm})(x_{pm} - \lambda) + (a_{pq^*} - w_{pq^*})x_{pq^*} + (a_{pq^*} - w_{(p)q^*}(min))\lambda$, where the weak inequality follows from (*).

Then, $\lambda u_{pm} \geq (a_{pq^*} - w_{(p)q^*}(min))\lambda$ and hence $u_{pm} + w_{(p)q^*}(min) \geq a_{pq^*}$. Then, $u_{p(q^*)}(min) + w_{(p)q^*}(min) \geq a_{pq^*}$, so $(u,w;x)$ is pairwise-strongly-stable and the proof is complete.■

By symmetry, this lemma holds if we reverse the roles between P-agents and Q-agents.

Lemma 4.2. *Let $(u,w;x)$ be a feasible allocation. Then $(u,w;x)$ is strongly-stable if and only if it is pairwise-strongly-stable.*

Proof. Suppose $(u,w;x)$ is strongly-stable. If condition (p) did not hold for some unsaturated pair (p,q), then buyer p and seller q could increase their earnings by transferring part of their labor time from some other partnership to $\{p,q\}$ (which is possible since (p,q) is unsaturated) and both players could profit from the increased earnings so obtained, which is absurd.

In the other direction, suppose by contradiction that $(u,w;x)$ satisfies (p) but it is not strongly-stable. This means that $(u,w;x)$ must be strongly-quasi-dominated by a feasible allocation $(u^*,w^*;x^*)$ via some coalition $R \cup T \neq \phi$, with $R \subseteq P$ and $T \subseteq Q$. By Definition 2.3.6 – (i_1), we have that

(1) $U_p < \sum_{q\in B(p,x^*)} u^*_{pq}\, x^*_{pq}$, for all $p\in R$ and $W_q < \sum_{p\in B(q,x^*)} w^*_{pq}\, x^*_{pq}$, for all $q\in T$.

Set $A \equiv \{(p,q) \in C(x^*);\ p\in R \text{ or } q\in T\}$, $D \equiv \{(p,q) \in C(x^*);\ p\in R,\ q\in T \text{ and } x_{pq}=0\}$ and $E \equiv \{(p,q) \in C(x^*);\ p\in R,\ q\in T \text{ and } x_{pq}>0\}$.

Adding up (1) yields

(2) $\sum_{p\in R} U_p < \sum_{p\in R}\sum_{q\in B(p,x^*)} u^*_{pq}\, x^*_{pq}$ and $\sum_{q\in T} W_q < \sum_{q\in T}\sum_{p\in B(q,x^*)} w^*_{pq}\, x^*_{pq}$.

Definition 2.3.6 – (i_2) implies that $[x_{pq} \geq x^*_{pq}$ and $u_{pq} = u^*_{pq}]$ for all $(p,q)\in A$ with $q \notin T$ and $[x_{pq} \geq x^*_{pq}$ and $w_{pq} = w^*_{pq}]$ for all $(p,q)\in A$ with $p \notin R$. Then,

(3) $\sum_{q\notin T}\sum_{p\in R\cap B(q,x^*)} u^*_{pq}\, x^*_{pq} + \sum_{p\notin R}\sum_{q\in T\cap B(p,x^*)} w^*_{pq}\, x^*_{pq}$

$= \sum_{q\notin T}\sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq} x^*_{pq} + \sum_{p\notin R}\sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq} x^*_{pq}$.

From (2) and (3) we get

$\sum_{p\in R}U_p + \sum_{q\in T} W_q < [\sum_{p\in R}\sum_{q\in T\cap B(p,x^*)} u^*_{pq}\, x^*_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*)} w^*_{pq}\, x^*_{pq}] + [\sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)} u^*_{pq}\, x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)} w^*_{pq}\, x^*_{pq}] = [(\sum_{p\in R}\sum_{q\in T\cap B(p,x^*) - B(p,x)} u^*_{pq}\, x^*_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*) - B(q,x)} w^*_{pq}\, x^*_{pq}) + (\sum_{p\in R}\sum_{q\in T\cap B(p,x^*)\cap B(p,x)} u^*_{pq}\, x^*_{pq} + \sum_{q\in T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w^*_{pq}\, x^*_{pq})] + [\sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq}x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq}x^*_{pq}] =$

(4) $\sum_{(p,q)\in D} (u^*_{pq} + w^*_{pq})x^*_{pq} + \sum_{(p,q)\in E} (u^*_{pq}+w^*_{pq})\, x^*_{pq} + \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq}x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq}x^*_{pq}$.

The feasibility of $(u^*,w^*;x^*)$ implies that condition (e) is satisfied, so the expression in (4) is equal to

$\sum_{(p,q)\in D} a_{pq}\, x^*_{pq} + \sum_{(p,q)\in E} a_{pq}\, x^*_{pq} + \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq}x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq}x^*_{pq}$.

From *(p)* and from the fact that all $(p,q)\in D$ are unsaturated, it follows that $\sum_{(p,q)\in D} a_{pq}\, x^*_{pq} \le \sum_{(p,q)\in D} (u_{p(q)}(min) + w_{(p)q}(min))\, x^*_{pq}$. From definition of E we have that $\sum_{(p,q)\in E} a_{pq}\, x^*_{pq} = \sum_{(p,q)\in E} (u_{pq}+w_{pq})\, x^*_{pq}$. Then,

(5) $\sum_{p\in R}U_p + \sum_{q\in T} W_q < \sum_{(p,q)\in D} (u_{p(q)}(min) + w_{(p)q}(min))\, x^*_{pq} + \sum_{(p,q)\in E} (u_{pq} + w_{pq})\, x^*_{pq} + \sum_{q\notin T} \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} u_{pq}x^*_{pq} + \sum_{p\notin R} \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} w_{pq}x^*_{pq}$.

We now have that for every $p\in R$,

$\sum_{q\in T\cap B(p,x^*)-B(p,x)} x^*_{pq} + \sum_{q\in T\cap B(p,x^*)\cap B(p,x)} x^*_{pq} + \sum_{q\in B(p,x^*)\cap B(p,x)-T} x^*_{pq} = r(p) = \sum_{q\in B(p,x) - B(p,x^*)} x_{pq} + \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} x_{pq} + \sum_{q\in B(p,x)\cap B(p,x^*)-T} x_{pq}$, so

$\sum_{q\in T\cap B(p,x^*)-B(p,x)} x^*_{pq} = \sum_{q\in B(p,x) - B(p,x^*)} x_{pq} + \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} (x_{pq} - x^*_{pq}) + \sum_{q\in B(p,x^*)\cap B(p,x)-T} (x_{pq} - x^*_{pq})$. Using that $u_{p(q)}(min) = u_p(min)$ for all $q\in T-B(p,x)$ we have

(6) $\sum_{q\in T\cap B(p,x^*)-B(p,x)} u_{p(q)}(min)\, x^*_{pq} = \sum_{q\in T\cap B(p,x^*)-B(p,x)} u_p(min)\, x^*_{pq} = \sum_{q\in B(p,x) - B(p,x^*)} u_p(min)\, x_{pq} + \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_p(min)\, (x_{pq} - x^*_{pq}) + \sum_{q\in B(p,x^*)\cap B(p,x)-T} u_p(min)(x_{pq} - x^*_{pq}) \le \sum_{q\in B(p,x) - B(p,x^*)} u_{pq}\, x_{pq} + \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq}\, (x_{pq} - x^*_{pq}) + \sum_{q\in B(p,x^*)\cap B(p,x)-T} u_{pq}\, (x_{pq} - x^*_{pq})$

Symmetrically, for every $q\in T$ we have that

(7) $\sum_{p\in R\cap B(q,x^*)-B(q,x)} w_{(p)q}(min) x^*_{pq} \le \sum_{p\in B(q,x)-B(q,x^*)} w_{pq}x_{pq} + \sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}(x_{pq} - x^*_{pq}) + \sum_{p\in B(q,x^*)\cap B(q,x)-R} w_{pq}(x_{pq} - x^*_{pq})$.

Adding up (6) and (7) yields

$\sum_{(p,q)\in D} (u_{p(q)}(min)+w_{(p)q}(min))\, x^*_{pq} \le [\sum_{p\in R}\sum_{q\in B(p,x) - B(p,x^*)} u_{pq}\, x_{pq} + \sum_{p\in R} \sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq}\, (x_{pq} - x^*_{pq}) + \sum_{p\in R} \sum_{q\in B(p,x^*)\cap B(p,x)-T} u_{pq}\, (x_{pq} - x^*_{pq})] + [\sum_{q\in T} \sum_{p\in B(q,x)-}$

$_{B(q,x^*)}\, w_{pq}x_{pq}) + \sum_{q\in T}\sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}(x_{pq} - x^*_{pq}) + \sum_{q\in T}\sum_{p\in B(q,x^*)\cap B(q,x)\text{-}R} w_{pq}(x_{pq}\text{-}x^*_{pq})]= [(\sum_{p\in R}\sum_{q\in B(p,x)\ -\ B(p,x^*)} u_{pq}\ x_{pq}+\sum_{p\in R}\ \sum_{q\in\ T\cap B(p,x)\cap B(p,x^*)} u_{pq}\ x_{pq} + \sum_{p\in R}\sum_{q\in B(p,x^*)\cap B(p,x)\text{-}T} u_{pq}\, x_{pq})] + [(\sum_{q\in T}\sum_{p\in B(q,x)\text{-}B(q,x^*)} w_{pq}x_{pq} +\sum_{q\in T}\sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}x_{pq} + \sum_{q\in T}\sum_{p\in B(q,x^*)\cap B(q,x)\text{-}R} w_{pq}x_{pq})] - [\sum_{p\in R}\sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq}x^*_{pq} + \sum_{p\in R}\sum_{q\in B(p,x^*)\cap B(p,x)\text{-}T} u_{pq}x^*_{pq} + \sum_{q\in T}\sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}\ x^*_{pq} + \sum_{q\in T}\sum_{p\in B(q,x^*)\cap B(q,x)\text{-}R} w_{pq}x^*_{pq}] = \sum_{p\in R}U_p + \sum_{q\in T} W_q - [\sum_{p\in R}\sum_{q\in T\cap B(p,x)\cap B(p,x^*)} u_{pq}\,x^*_{pq} + \sum_{q\in T}\sum_{p\in R\cap B(q,x^*)\cap B(q,x)} w_{pq}\, x^*_{pq}] - [\sum_{q\notin T}\sum_{p\in R\cap\ B(q,x^*)\cap\ B(q,x)} u_{pq}x^*_{pq} + \sum_{p\notin R}\sum_{q\in T\cap\ B(p,x^*)\ \cap\ B(p,x)} w_{pq}x^*_{pq}] = \sum_{p\in R}U_p + \sum_{q\in T} W_q - \sum_{(p,q)\in E} (u_{pq} + w_{pq})\ x^*_{pq} - \sum_{q\notin T}\sum_{p\in R\cap\ B(q,x^*)\cap\ B(q,x)} u_{pq}x^*_{pq} - \sum_{p\notin R}\sum_{q\in T\cap\ B(p,x^*)\ \cap\ B(p,x)} w_{pq}x^*_{pq}$. Then,

$\sum_{(p,q)\in D} (u_{p(q)}(min)+w_{(p)q}(min))\ x^*_{pq} \le \sum_{p\in R}U_p + \sum_{q\in T} W_q - \sum_{(p,q)\in E} (u_{pq} + w_{pq})\ x^*_{pq} - \sum_{q\notin T}\sum_{p\in R\cap\ B(q,x^*)\cap\ B(q,x)} u_{pq}x^*_{pq} - \sum_{p\notin R}\sum_{q\in T\cap\ B(p,x^*)\ \cap\ B(p,x)} w_{pq}x^*_{pq}$, so

$\sum_{p\in R}U_p + \sum_{q\in T} W_q \ge \sum_{(p,q)\in D} (u_{p(q)}(min)+w_{(p)q}(min))\ x^*_{pq} + \sum_{(p,q)\in E} (u_{pq} + w_{pq})\ x^*_{pq} + \sum_{q\notin T}\sum_{p\in R\cap\ B(q,x^*)\cap\ B(q,x)} u_{pq}x^*_{pq} + \sum_{p\notin R}\sum_{q\in T\cap\ B(p,x^*)\ \cap\ B(p,x)} w_{pq}x^*_{pq}$, which contradicts (5). Hence *(u,w;x)* is strongly-stable. ■

It follows immediately from these two lemmas that:

Theorem 4.3. *Let (u,w;x) be a feasible allocation. The following assertions are equivalent*

(i_1) (u,w;x) is strongly-stable;

(i_2) (u,w;x) is pairwise-strongly-stable;

(i_3) for all $p\in P$ and feasible labor allocation x' we have that

(*) $U_p \ge \sum_{q\in B(p,x')}(a_{pq}\text{-}w'_{pq})x'_{pq}$,

where $w'_{pq}x'_{pq} = w_{pq}x'_{pq}$ *if* $x_{pq} \ge x'_{pq}$, $w'_{pq}x'_{pq} = w_{pq}x_{pq} + w_{(p)q}(min)(x'_{pq}\text{-}x_{pq})$ *if* $0<x_{pq}<x'_{pq}$ *and* $w'_{pq}x'_{pq} = w_{(p)q}(min)x'_{pq}$ *if* $0=x_{pq}< x'_{pq}$.

The proof of the existence theorem uses Theorem 4.5 and Proposition 4.8, which needs Lemma 4.7 below. For simplicity of notation, in what follows, we will use some times $\sum_P$, $\sum_Q$, $\sum_{PxQ}$ to denote, respectively, $\sum_{p\in P}$, $\sum_{q\in Q}$, $\sum_{(p,q)\in PxQ}$, and so on.

Lemma 4.7. *Let (u,w;x) be a strongly-stable allocation. Then x is an optimal labor time allocation.*

Proof. Let x' be any feasible labor time allocation. For all $(p,q)\in P\times Q$, let $\Delta_{pq}= x_{pq} - x'_{pq}$. We must show

(1) $\sum_{PxQ} a_{pq}\, \Delta_{pq} \geq 0.$

Define $T\equiv\{(p,q)\in C(x);\ x_{pq}-x'_{pq}\geq 0\}$, $T^*\equiv\{(p,q)\in C(x);\ x_{pq}-x'_{pq} <0\}$, $T(p)\equiv\{q;\ (p,q)\in T\}$, $T^*(p)\equiv\{q;\ (p,q)\in T^*\}$, $T(q)\equiv\{p;\ (p,q)\in T\}$ and $T^*(q)\equiv\{p;\ (p,q)\in T^*\}$. Then,

(2) $\sum_{q\in T(p)} \Delta_{pq} + \sum_{q\in T^*(p)} \Delta_{pq} =0$ for all $p\in P$ and $\sum_{p\in T(q)} \Delta_{pq} + \sum_{p\in T^*(q)} \Delta_{pq} =0$ for all $q\in Q$, by feasibility of x and x'. Set

(3) $u_p\equiv min\{ u_{pq};\ q\in T(p)\}$ and $w_q\equiv min\{ w_{pq};\ p\in T(q)\}$.

Then, $\sum_{C(x)} a_{pq}\, \Delta_{pq} = \sum_{T} a_{pq}\, \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} = \sum_{T}(u_{pq} + w_{pq})\, \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq}$
$= \sum_{P} \sum_{q\in T(p)} u_{pq}\, \Delta_{pq} + \sum_{Q}\sum_{p\in T(q)} w_{pq}\, \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} \geq \sum_{P} u_p \sum_{q\in T(p)} \Delta_{pq} +$
$+\sum_{Q} w_q \sum_{p\in T(q)} \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} = - \sum_{P} u_p \sum_{q\in T^*(p)} \Delta_{pq} - \sum_{Q} w_q \sum_{p\in T^*(q)} \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq}$
$= \sum_{T^*} - (u_p + w_q)\, \Delta_{pq} + \sum_{T^*} a_{pq}\, \Delta_{pq} = \sum_{T^*} (a_{pq} - (u_p + w_q))\, \Delta_{pq}$, where the third last equality follows from (2).

Now, let $(p,q)\in T^*$. We have $x_{pq} < x'_{pq}$, so $x_{pq}<r(p)$ and $x_{pq}< s(q)$ and then (p,q) is unsaturated. From (3) $u_p=u_{pm}$ for some $m\in T(p)$ and $w_q= w_{kq}$ for some $k\in T(q)$, so $m\neq q$ and $k\neq p$ because $(p,q)\in T^*$, so $u_p\geq u_{p(q)}(min)$ and $w_q\geq w_{(p)q}(min)$. By strong stability, $(a_{pq} - (u_{p(q)}(min) + w_{(p)q}(min))) \leq 0$, so $a_{pq}-(u_p+w_q)\leq 0$. We also have $\Delta_{pq}<0$. Therefore, $\sum_{T^*} (a_{pq} - (u_p + w_q))\, \Delta_{pq}\geq 0$ and so (1) is proved.■

Note that if x is an optimal labor time allocation and $(u,w;x')$ is a strongly stable allocation with $x'\neq x$, then x is not necessarily compatible with (u,w). This is because u and w are not indexed according to x.

Proposition 4.8. *The set of non-discriminatory strongly-stable allocations coincides with the set of dual allocations.*

Proof. Let $(u,w;x)$ be a dual allocation. The definition of u and w implies that $(u,w;x)$ is non-discriminatory. This allocation is feasible by (D) and by the construction of $(u,w;x)$. Property *(p)* is implied by (C) and (D). Theorem 4.3 then implies that $(u,w;x)$ is strongly-stable.

Conversely, let $(u,w;x)$ be a non-discriminatory strongly-stable allocation. Define (y,z) such that $y_p=u_p(min)$ and $z_q=w_q(min)$ for all $p\in P$ and $q\in Q$. Then

(1) $u_{pq}=u_p(\text{min})=y_p$ and $w_{pq}=w_q(\text{min})=z_q$ for all $(p,q)\in C(x)$.

Lemma 4.7 implies that x is an optimal labor time allocation, so it is an optimal solution of (P1). Theorem 4.3 implies that *(p)* is satisfied, so $y_p+z_q \geq a_{pq}$ if $x_{pq}=0$. On the other hand, from (1) it follows that $y_p + z_q=a_{pq}$ if $x_{pq}>0$. Therefore, (B2) is satisfied. The feasibility of *(u,w;x)* implies both: *(y,z)* minimizes (B1) and (B3) is satisfied. Hence *(u,w;x)* is a dual allocation and the proof is complete. ■

Theorem 4.9. *The set of competitive equilibrium allocations, the set of stable allocations for the flexible market, the set of stable allocations for the rigid market and the core are always non-empty.*

Proof. By Theorem 4.6, it is enough to show that the set of competitive equilibrium allocations is non-empty. Theorem 4.5 implies that the set of competitive equilibrium allocations contains the set of non-discriminatory strongly-stable allocations. Proposition 4.8 shows that this set is precisely the set of dual allocations, so it is always non-empty by the Duality Theorem. Hence, the set of competitive equilibrium allocations is always non-empty, and the proof is complete. ■

For the proof of Theorem 5.2 we need Lemma 5.1.

Lemma 5.1. *Let (u,w;x) and (u',w';x) be strongly-stable allocations. Then* $(u^*,v_*;x)$ *and* $(u_*,v^*;x)$*, defined in (u1) and (u2), are strongly-stable allocations.*

Proof. It is clear that $(u^*,w_*;x)$ is feasible. Also, $u^*_{p(q)}(min) \geq u_{p(q)}(min)$, $u^*_{p(q)}(min) \geq u'_{p(q)}(min)$ and $w_{*(p)q}(min) = min\{w_{(p)q}(min),w'_{(p)q}(min)\}$ (suppose $w_{p'q}= w_{(p)q}(min)=min\{w_{(p)q}(min),w'_{(p)q}(min)\}$. Then, $w_{p'q} \leq w_{p''q}$ and $w_{p'q} \leq w'_{p''q}$ for all $p''\in B(q,x)-\{p\}$, so $w_{p'q} \leq w'_{p'q}$, so $w_{*p'q}=w_{p'q} \leq w_{*p''q}$ for all $p''\in B(q,x)-\{p\}$, and then $w_{*(p)q}(min)= w_{(p)q}(min)$). Suppose $\{p,q\}$ is unsaturated and $w_{*(p)q}(min)=w_{(p)q}(min)$. We have that $u^*_{(p)q}(min) + w_{*(p)q}(min)= u^*_{p(q)}(min) + w_{(p)q}(min) \geq u_{p(q)}(min) + w_{(p)q}(min) \geq a_{pq}$ from strong stability of $(u,w;x)$. Then, $u^*_{(p)q}(min) + w_{*(p)q}(min) \geq a_{pq}$ for all unsaturated pair $\{p,q\}$. Hence $(u^*,w_*;x)$ is strongly-stable. With symmetric arguments to those used above, it can be shown that $(u_*,w^*;x)$ is also strongly-stable.■

Theorem 5.2. *Let x be an optimal labor time allocation. Then,*

a) the set of the strongly-stable allocations compatible with x is a complete lattice under both partial orders $\geq_P$ and $\geq_Q$;

b) this set has a P-optimal and a Q-optimal allocations and *c) properties (x1) and (x2) hold.*

Proof. a) It is immediate from Lemma 5.1 and the fact that the strongly-stable money allocations compatible with x is a compact set of some Euclidean space; b) and c) follow from a). ■

Theorem 5.3. *Let x be an optimal labor time allocation. Then,*

a) the set $A(x)$ is a complete lattice under both partial orders $\geq_P$ and $\geq_Q$;

b) there always exist the P-optimal and Q-optimal competitive equilibrium allocations for $A(x)$ and

c) properties (x1) and (x2) hold.

Proof. It is immediate from Theorem 5.2, due to the fact that competitive equilibrium allocations are strongly-stable, and the meet and joint of two allocations that are non-discriminatory for one of the sides are still non-discriminatory for that side. ■

Proposition 5.4. *Let $(u,w;x)$ be a competitive equilibrium allocation in $A(x)$ and let x' be an optimal labor time allocation. Set $f_{x'}(u,w;x)\equiv(u',w';x')$. Then, $(u',w';x')$ is a competitive equilibrium allocation in $A(x')$. Furthermore, $U_p=U'_p$ for all $p\in P$ and $W_q=W'_q$ for all $q\in Q$.*

Proof. Theorem 4.5 implies that $(u,w;x)$ is a P-non-discriminatory strongly stable allocation. Then, from Corollary 4.4, we have that $U_p\geq\sum_{q\in B(p,x')}(a_{pq}-w_{pq})x'_{pq}$ for all $p\in P$. Therefore,

(1) $U_p\geq U'_p$ for all $p\in P$.

By definition of $f_{x'}$, $w'=w$ and u' is feasibly defined, so $W_q=W'_q$ for all $q\in Q$ and

(2) $(u',w';x')$ is feasible and P-non-discriminatory.

By (1) and the feasibility of the two allocations we get that

(3) $\sum_{PxQ} a_{pq}x_{pq} = \sum_P U_p + \sum_Q W_q \geq \sum_P U'_p + \sum_Q W'_q = \sum_{PxQ} a_{pq}x'_{pq}$.

Since x' is optimal we must have equality in (3), so equality in (1). Then, $U=U'$ and, by Corollary 4.4,

(4) $U'_p \geq \sum_{q\in B(p,x'')}(a_{pq}-w_{pq})x''_{pq}$ for all $p\in P$ and feasible labor time allocation x''.

It follows from (2) and (4) that Corollary 4.4 applies for $(u',w;x')$, so $(u',w;x')$ is P-non- discriminatory strongly stable. Theorem 4.5 then implies that $(u',w;x')$ is competitive. Hence $f_{x'}(u,w;x)\in A(x')$ and the proof is complete. ■

Theorem 5.5. *Let $(u,w;x)$ be the P-optimal(respectively, Q-optimal) competitive equilibrium allocation for $A(x)$. Let x' be any optimal labor time allocation. Then, $f_{x'}(u,w;x)$ is the P-optimal(respectively, Q-optimal) competitive equilibrium allocation for $A(x')$.*

Proof. We are going to show the first assertion. The second one follows dually. Then set $f_{x'}(u,w;x)\equiv(u',w;x')\in A(x')$. Let $(u'',w'';x')$ be in $A(x')$ and set $f_x(u'',w'';x')\equiv(u^*,w'';x)\in A(x)$. Proposition 5.4 and the *P*-optimality of $(u,w;x)$ imply that $U'_p=U_p\geq U^*_p=U''_p$ for all $p\in P$, so $U'_p\geq U''_p$ for all $p\in P$. Hence, $(u',w;x')$ is the *P*-optimal competitive equilibrium allocation for $A(x')$, which completes the proof. ■

Theorem 5.6. *Let $(\bar{u},\underline{w};x)$ and $(\bar{u}',\underline{w}';x')$ be the P-optimal competitive equilibrium allocations of $A(x)$ and $A(x')$, respectively. Then, $\bar{U}=\bar{U}'$ and $\underline{W}=\underline{W}'$.*

Proof. By Theorem 5.5, $f_{x'}(\bar{u},\underline{w};x)=(\bar{u}',\underline{w}';x')$. Now use the definition of $f_{x'}$ and Proposition 5.4. ■

APPENDIX II: COMPETITIVE MARKET WITH NON-DISCRIMINATORY DEMANDS

In this section we define a demand correspondence which applies to the assignment game and to the time-sharing assignment game. The resulting competitive market will be called *competitive market with non-discriminatory demands.* The competitive market defined in section 3.1 will be referred here as *competitive market with discriminatory demands.*

Specifically, in the *competitive market with non-discriminatory demands*, each seller q supplies $s(q)$ u.l.t. of type q (we identify seller q with the type of the u.l.t. supplied by him). Then the set of all types can be denoted by Q. The prices of all types are announced. Buyers have preferences over the u.l.t. supplied by the sellers at the given prices. Buyer p will demand bundles of types of u.l.t. that are feasible for him (that respect his quota). Furthermore, in any demanded bundle, every type whose number of units in the bundle is positive maximizes p's individual surpluses.

Under a competitive equilibrium the bundle of goods allocated to buyer p is a feasible assignment vector for p and it belongs to the demand set of the buyer at the given prices. Thus, for the purpose of analyzing competitive equilibria, there will be no loss in restricting the demand set of a buyer to bundles of goods that are feasible assignment vectors for the buyer.

Therefore, in the competitive market with non-discriminatory demands, given a price vector π, each buyer $p \in P^*$ will demand the feasible assignment vector x_p if, for all $q \in Q$ with $x_{pq}>0$, and for all $q' \in Q$, we have that $(a_{pq}-\pi_q) \geq (a_{pq'}-\pi_{q'})$. Thus, buyer p will get equal surpluses with all $q \in Q$ with $x_{pq}>0$.

Set $ND_p(\pi)$ the demand set of buyer $p \in P^*$ at prices π in the competitive market with non-discriminatory demands. That is,

$$ND_p(\pi) \equiv \{x_p \in X_p;\ (a_{pq}-\pi_q) \geq (a_{pq'}-\pi_{q'})\ \ \forall\ q \in Q \text{ with } x_{pq}>0 \text{ and } q' \in Q\}.$$

From this definition, if $x_p \in ND_p(\pi)$ and $x_{p0}>0$ then buyer p gets a zero individual surplus with all $q \in Q$ with $x_{pq}>0$. In this case, $(a_{pq}-\pi_q) \leq 0$ for all $q \in Q$. Therefore, if buyer p gets a positive surplus with some $q \in Q$, then we must have that $x_{p0}=0$ for all $x_p \in ND_p(\pi)$. In this case, if the amount of units of the types which maximize p's individual surpluses is not enough to fill the quota of buyer p, the set $ND_p(\pi)$ will be empty.

REMARK A.1. Clearly, $ND_p(\pi) \subseteq D_p(\pi)$. Thus, every competitive equilibrium for the competitive market with non-discriminatory demands is a competitive equilibrium for the competitive market with discriminatory demands. Furthermore, if $x_p \in D_p(\pi)$ and $(a_{pq}-\pi_q) = (a_{pq'}-\pi_{q'})$, $\forall q, q' \in Q$ with $x_{pq}>0$ and $x_{pq'}>0$, then $x_p \in ND_p(\pi)$. Therefore, if a competitive equilibrium for the competitive market with discriminatory demands is a non-discriminatory allocation, then the allocation is a competitive equilibrium for the competitive market with non-discriminatory demands.

It is also clear that under the rules of the assignment game, for all $p \in P^*$, $x_{pq} \in \{0,1\}$ for all $q \in Q$ and $\sum_{q \in Q} x_{pq}=1$ for all $p \in P^*$. Then, $ND_p(\pi) = D_p(\pi) \neq \phi$ in that market.■

Clearly, both competitive markets coincide in the assignment game. However, from our previous results and Theorem A.1 below, we can conclude that the equivalence observed in the assignment game between the set of dual allocations and the core, as well as that between the set of dual allocations and the set of competitive equilibrium allocations under discriminatory demands, is less robust to the introduction of time into the assignment game than the equivalence between the set of dual allocations and the set of competitive equilibrium allocations under non-discriminatory demands. In the time-sharing assignment game, the set of dual allocations coincides no longer with the core or the set of competitive equilibrium allocations under discriminatory demands. But it still is the set of competitive equilibrium allocations under non-discriminatory demands.

Theorem A.1. *Let σ be a feasible allocation. Then σ is a competitive equilibrium allocation for the market with non-discriminatory demands if and only if it is a dual allocation.*

Proof. In fact, it follows from Remark A.1 and Theorem 4.5 that if σ is a competitive equilibrium allocation for the competitive market with non-discriminatory demands then it is strongly-stable and no *Q*-agent discriminates any *P*-agent. Since, at σ, each buyer receives the same payoffs at all individual trades, we have that σ is a non-discriminatory strongly-stable allocation, and so it is a dual allocation by Proposition 4.8. Conversely, if σ is a dual allocation then it is a non-discriminatory strongly-stable allocation by Proposition 4.8, then it is *P*-non-discriminatory strongly-stable and so Theorem 4.5 implies that it is a competitive equilibrium allocation for the market with discriminatory demands. Since σ is a non-discriminatory allocation, it follows from Remark A.1 that σ is also a competitive equilibrium allocation for the market with non-discriminatory demands. Then, the competitive equilibrium allocations for the market with non-discriminatory demands are precisely the dual allocations. ■

Hence, in the time-sharing assignment game, the equivalence between dual allocations and competitive equilibrium allocations is preserved in the *competitive market with non-discriminatory demands* and it is not in the competitive market with discriminatory demands.

Theorem A.2. *The set of competitive equilibrium money allocations for the market with non-discriminatory demands is a complete lattice under $\geq_P$ and $\geq_Q$.*

Proof. The set of dual allocations is the intersection of the set of *P*-non-discriminatory strongly-stable allocations with the set of *Q*-non-discriminatory strongly-stable allocations. Theorem 5.3 implies that the restriction of these two sets to a given optimal labor time allocation are complete lattices. Then, the set of dual allocations compatible with an optimal labor time allocation is the intersection of two complete lattices, so it is also a complete lattice. As remarked before, the dual money allocations are compatible with any optimal labor time allocation. Then, $\geq_P$ and $\geq_Q$ are also partial orders for the set of dual **money** allocations. (Observe that these binary relations are not partial orders for the set of dual allocations, since they do not satisfy the anti-symmetric property). Hence, **the whole set of dual money allocations is a lattice under** $\geq_P$ **and** $\geq_Q$**.** ■

Therefore, the complete lattice property of the core money allocations and of the set of the competitive equilibrium money allocations under discriminatory demands, observed in the assignment game, is less robust to the introduction of time into that model than the lattice property of the set of the competitive equilibrium money allocations under non-discriminatory demands.

Theorem A.3 asserts that the competitive equilibrium allocations under non-discriminatory demands are given by the Cartesian product of the corresponding set of money allocations by the set of optimal labor time allocations. Then this property applied to the assignment game is more robust to the introduction of time into that model than the property concerning the competitive equilibrium allocations under discriminatory demands.

Theorem A.3. *Let* $(u,w;x)$ *be a competitive equilibrium allocation under non-discriminatory demands. Then,*

(a) x *is an optimal labor time allocation, and*

(b) if x' *is an optimal labor time allocation, then* $(u,w;x')$ *is a competitive equilibrium allocation under non-discriminatory demands.*

Proof. (a) follows from Lemma 4.7, Proposition 4.8 and Theorem A.1. For part (b) uses Theorem A.1 and the fact that every dual solution of P1* is compatible with any optimal solution of P1. ■

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