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On Simple Outcomes and Cores

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JEL Codes: C78; D78.

ON SIMPLE OUTCOMES AND CORES

by

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ABSTRACT

For a general coalitional game with non-transferable utility (NTU game) and a finite set of players, *(N,V),* Scarf (1967) proved that every balanced game has a non-empty core. Billera (1970) showed, through an example, that this condition is not always necessary when *V(N)* has a supremum. By using the concepts of simple outcome and Pareto simple outcome, the present paper provides a weaker condition than balancedness, which is sufficient for the non-emptiness of the core in the general case and is necessary when *V(N)* has a supremum. It is also necessary for any TU game. Our proof avoids the use of balancedness and specialized mathematical tools. Instead, it is elementary and only employs simple combinatorial arguments.

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INTRODUCTION

The present study represents part of a continuing investigation of the properties of a group of individually rational and feasible outcomes, called *simple outcomes* (the definition is given in the text), aiming to deal with the existence problem of core outcomes. In this paper we approach the general coalitional game with non-transferable payoff (NTU game for short) and finite number of players, whose formulation is presented by Kannai in the Handbook of Game Theory with Economic applications, vol. 1, edit. by Aumann and Hart. For this game, every blocking coalition of a simple outcome, if any, has a sub-blocking coalition formed with agents whose payoffs are zero. There is a *stable coalition,* that is, a coalition that does not have any sub-blocking coalition. In addition, if it is non-empty then it is effective for the outcome. Clearly, the core is a subset of the set of simple outcomes.

Scarf (1967), through the theory of balanced games, proves that the core of the general NTU game is non-empty if the game is balanced. Billera (1970) presents an example of a game (N,V) in which the core is non-empty but the balancedness condition is not satisfied. Therefore, unlike the coalitional games with transferable utility (TU for short), the set of balanced games is smaller than the set of games with non-empty core.

The present paper provides a novel sufficient condition, which is weaker than balancedness, for the non-emptiness of the core in general NTU games. This condition turns out to be also necessary for a large class of games that includes the game of Billera's example. To understand the condition, call a game *singular* if the set of simple outcomes is non-empty and no simple outcome out of the core is Pareto optimal among all simple outcomes. We prove that e*very singular game has a non-empty core.* Moreover, i*f the core of (N,V) is non-empty and $V(N)$ has a supremum, then the game is singular.*

That the condition is sufficient follows from the fact that the set of simple payoff vectors is compact and, when the game is singular, it is also non-empty. This guarantees the existence of a Pareto optimal simple outcome.

The proof that the singularity condition is necessary is not straightforward, but it is short and only uses elementary combinatorial arguments.

Clearly, if $V(N)$ has a supremum then the set of balanced games is properly contained in the set of singular games, so to be singular is a weaker condition than to be balanced. When $V(N)$ does not have a supremum, an example in this paper shows that we may have non-empty cores in non-singular and non-balanced games. However, the singularity of a game is a necessary and sufficient condition for the non-emptiness of the core in a quite large class of NTU games, for which $V(N)$ does not have a supremum. These are the TU games:[2] *The core of a TU game is non-empty if and only if it is singular.*

The market games, commonly used to model exchange economies, compose a very special group of NTU games. In order to illustrate the use of our results we show that the core of the exchange economy of Shapley and Scarf (1974) is non-empty, by proving that the corresponding market game is singular.[3]

A section with examples illustrates the concepts and motivates the theorems.

This paper is organized as follows. In section 2 we describe the game and the conceptual framework to be used in the other sections. Section 3 presents the illustrative examples. In section 4 we state and proof the theorems. Section 5 is devoted to an economic application. Section 6 concludes the paper and discusses related work.

2. FRAMEWORK

The following description follows the lines of Ch. 12, written by Y. Kannai, of Handbook of Game Theory with Economic applications, vol.1, edit. by Aumann and Hart. There is a finite set of players $N=\{1,2,\ldots,n\}$. An *outcome* is a vector $x=(x_1,\ldots,x_n)\in R^n$. We can interpret x_i as the amount paid to the ith player. Associated to any subset (coalition) S of N there is a set $V(S)$ such that:

$$V(\phi)=\phi, \qquad \text{(A.1)}$$

for all $S\neq\phi$, $V(S)$ is a non-empty closed subset of R^N, (A.2)

if $x\in V(S)$ and $y_i\leq x_i$ for all $i\in S$, then $y\in V(S)$. (A.3)

We will always assume that there exists a closed set $F\subset R^N$ such that

[2] The proof of this result is presented in Sotomayor (2005-b). For the sake of completeness it is sketched in section 5.

$V(N)=\{x\in R^N; \exists y\in F \text{ with } x_i\leq y_i \text{ for all } i\in N\}$. (A.4)

Thus, a payoff x is *feasible* if $x\in V(N)$. It is *individually rational* if for every $i\in N$ and every $y\in V(\{i\})$, $x_i\geq y_i$. For simplification we will assume that

$V(\{i\})=\{x\in R^N; x_i\leq 0\}$. (A.5)

Denote $R^N_+=\{x\in R^N; x_i\geq 0 \text{ for all } i\in N\}$. Then, R^N_+ is the set of individually rational payoff vectors and $V(N)\cap R^N_+$ is the set of feasible and individually rational payoff vectors. Feasible, individually rational payoff vectors exist only if F contains at least some vectors with non-negative components. We will assume that

$F\cap R^N_+$ is a non-empty compact set. (A.6)

Consequently, there exists $y\in F$ with $y_i\geq 0$ for all $i\in N$, so $\boldsymbol{(0,...,0)\in V(N)}$. Note that if $x\in V(N)\cap R^N_+$, there is some $y\in F\cap R^N_+$ with $0\leq x_i\leq y_i$ for all $i\in N$. Since $F\cap R^N_+$ is bounded, there exists $M>0$ such that $y_i\leq M$ for all $i\in N$, and so $x_i\leq M$ for all $i\in N$. Hence, **$V(N)\cap R^N_+$ is non-empty and bounded from above.**

Definition 1. *We say that coalition* **S blocks the outcome x via y** *if $y\in V(S)$ and $x_i<y_i$ for all $i\in S$. The core of the game (N,V) is the set of all feasible payoff vectors that are not blocked by any coalition.*

Remark 1. a) By (A.3), S blocks x if and only if $x\in Int V(S)$. Hence, **the core of the game V coincides with $V(N)-\cup_{S\subseteq N} Int\, V(S)$.**

b) Let z be a payoff vector such that $z_i\leq x_i$ for all $i\in S$. Then, if S blocks x via y, $y\in V(S)$ and $z_i\leq x_i<y_i$ for all $i\in S$, so $z_i<y_i$ for all $i\in S$, and so S blocks z via y. Thus, **if $x\in Int\, V(S)$ then $z\in Int\, V(S)$.**

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[3] A direct proof, without passing through the market game, is given in Sotomayor (2005-d).

Given a feasible payoff vector x, *a stable coalition for* x is any coalition $T(x)$ such that (a) no subset of it can block x; (b) if $i \notin T(x)$ then i is part of some blocking coalition of x and (c) if S blocks x via y then $S\text{-}T(x)$ also blocks x via y', where $y'_i \geq y_i$ for all $i \in S\text{-}T(x)$. (The idea is that the agents in $S\text{-}T(x)$ can do better trades among them rather than with the agents of $T(x)$). It then follows that **if $i \notin T(x)$ then i is part of some blocking coalition of x contained in $N\text{-}T(x)$.** Of course, $T(x)$ is not uniquely defined and may be empty.

Definition 2. Let x *be an outcome and let* $T(x)$ *be a stable coalition for* x*. The outcome* x *is* ***simple via T(x)*** *(or* ***simple*** *for short) if (i))* $x \in V(N) \cap R^N_+$*, (ii) in case* $T(x) \neq N$ *then* $x_i = 0$ *for all* $i \notin T(x)$ *and (iii) in case* $T(x) \neq \phi$ *then* $x \in V(T(x))$*.*

Clearly, any core payoff vector is simple. In this case N is the only stable coalition. If $x=(0,\ldots,0) \in IntV(N)$, the set of simple payoff vectors is non-empty, since x is simple via $T(x)=\phi$. Also, if $x=(0,\ldots,0) \in V(S)$ for all coalitions S (that is, every coalition is effective for x), then x is simple. In fact, set $A \equiv \{i \in N;\ if\ i \in S\ \text{then}\ x \notin IntV(S)\}$. Suppose $A=\phi$. Then *i* belongs to some blocking coalition of x, for every $i \in N$. Then, $T(x)=\phi$ is a stable coalition for x and x is simple. Now, suppose $A \neq \phi$. Clearly, every $i \notin A$ belongs to some blocking coalition and if S blocks x then $S \subseteq N\text{-}A$. Then, $T(x)=A$ is a stable coalition for x. Now, use that $x \in V(T(x))$ to get that x is simple.

We must point out that the stable coalitions are the main ingredient of a simple outcome. The idea behind a simple outcome x is that a stable coalition $T(x)$ is formed when agents act cooperatively and succeed in their interactions. If $T(x)=\phi$ is the only stable coalition, then no cooperative interaction exists. If we have an algorithm to find $T(x)$ then we have a procedure to find a simple outcome and, consequently, a core outcome when it exists.

Definition 3. *The simple payoff vector* x^* ***extends*** *the simple payoff vector* x *if* $x^*_i \geq x_i$ *for all* $i \in N$ *with strict inequality for at least one* $i \in N$*.*

Definition 4. *The payoff vector* x *is a* ***Pareto optimal simple payoff vector*** *(PS for short) if it is simple and does not have any simple extension.*

Therefore, if x is PS and $y_i>x_i$ for some player i and some simple payoff vector y, then there exists some other player k such that $y_k<x_k$.

Definition 5. *We say that the game* (N,V) *is* ***singular*** *if (i) the set of simple payoff vectors is non-empty and (ii) every simple payoff vector out of the core (if any) has a simple extension.*

Then, if every simple outcome is in the core the game is singular.

3. EXAMPLES

The following examples illustrate the concepts introduced in the previous section and motivate the results of the next section.

Example 1. (A non-singular and non-balanced game with an empty core) Consider $N=\{1,2,3\}$; $V(S)=\{x=(x_1,x_2,x_3)\in R^3;\ \sum_{i\in S} x_i\leq 1\}$, if $|S|\geq 2$; $V(i)=\{x=(x_1,x_2,x_3)\in R^3;\ x_i\leq 0\}$, if $i=1,2,3$.

The core of this game is empty because every feasible and individually rational outcome belongs to $IntV(i,j)$ for some i and j. We claim that the game is not singular. In fact, the outcome $(0,0,0)$ is simple, since it is in $IntV(N)$. It is the only simple outcome for this game. In fact, let x be a simple outcome. Let $T(x)$ be a stable coalition for x. Set $S(x)\equiv N\text{-}T(x)$. Individual rationality plus feasibility of x imply $x_1 + x_2 + x_3\leq 1$ and $x_i\geq 0$ for all $i=1,2,3$. Since the game has an empty core, $T(x)\neq N$.

We have that $|T(x)|\neq 2$, for otherwise $S(x)=\{k\}$, for some $k=1,2,3$, so k must be in some blocking coalition $S'\subseteq S(x)$, so $\{k\}$ blocks x, which is a contradiction. Thus, the possibilities for $T(x)$ and $S(x)$ are:

1) $T(x)=\phi$, $S(x)=\{1,2,3\}$. Then $x=(0,0,0)$.

2) $T(x)=\{i\}$, $S(x)=\{j,k\}$. Definition 2-(iii) requires that $x\in V(i)$, so $x_i=0$. On the other hand, $x_j=x_k=0$. Hence, $x=(0,0,0)$.

Therefore, the only simple outcome is $x=(0,0,0)$. Hence, this game is not singular, since x cannot be extended to another simple outcome. Theorem 1 confirms that this fact is not accidental: **Every game with an empty core is not singular.** This game is not balanced, since it has an empty core. ■

Example 2. (a non-singular and non-balanced game with empty core) Consider $N=\{1,2,,3,4\}$; $V(N)=\{x=(x_1,x_2,x_3,x_4)\in R^4;\ x_1\leq 0,\ x_2\leq 0,\ x_3\leq 1/2,\ x_4\leq 1/2\}$, $V(1,2)=\{x=(x_1,x_2,x_3,x_4)\in R^4;\ x_1\leq 1/4,\ x_2\leq 1\}$, $V(3,4)=\{x=(x_1,x_2,x_3,x_4)\in R^4;\ x_3\leq 1/2,\ x_4\leq 1/2\}$, $V(S)=\{\ x=(x_1,x_2,x_3,x_4)\in R^4;\ x_i\leq 0$ for all $i\in S\}$, for $S\neq\{1,2\}$, $S\neq\{3,4\}$ and $S\neq N$.

Individual rationality plus feasibility of x imply that $x_1=x_2=0,\ 0\leq x_3\leq 1/2,\ 0\leq x_4\leq 1/2$. The outcome $x=(0,0,1/2,1/2)$ is simple and it is not in the core. In fact, $y=(1/4,1,0,0)$ dominates x via $\{1,2\}$. Now, observe that the only blocking coalition of x is $\{1,2\}$. Then, no sub coalition of $\{3,4\}$ blocks x. By the definition of $T(x)$, it follows that $T(x)=\{3,4\}$ is a stable coalition for x (and the only one). Also, $x\in V(34)$ and $x_1=x_2=0$, so x is simple.

In this game, $z=(0,0,0,0)\in V(S)$ for all coalitions S, so z is simple. Outcome y also dominates z, so z is not in the core. The outcome z can be extended to the simple outcome x. However, x cannot be extended to a simple outcome. Hence, the game is not singular. Then x is a PS outcome out of the core. The game is not balanced, since $y\notin VN)$, $y\in V(12)\cap V(34)$ and the collection $\{\ \{1,2\},\{3,4\}\}$ is balanced.

It is a matter of verification that the core is empty. ■

The fact that the core of Example 2 is empty is not implied by the non-balancedness of the game. According to Theorem 2, **the core is empty because the game is not singular and $V(N)$ has a supremum.**

If $V(N)$ does not have a supremum, we may have a non-empty core in non-singular and non-balanced games. Also a PS outcome may be out of the core in games with non-empty cores. See Example 3, which differs from Example 2 only in the set $V(N)$.

Example 3. (a non-singular and non-balanced game with non-empty core) Consider $N=\{1,2,,3,4\}$; $V(N)=\{x=(x_1,x_2,x_3,x_4)\in R^4;\ \sum_{i\in N} x_i\leq 1\}$, $V(1,2)=\{x=(x_1,x_2,x_3,x_4)\in R^4;\ x_1\leq 1/4, x_2\leq 1\}$, $V(3,4)=\{x=(x_1,x_2,x_3,x_4)\in R^4;\ x_3\leq 1/2,\ x_4\leq 1/2\}$, $V(S)=\{\ x=(x_1,x_2,x_3,x_4)\in R^4;\ x_i\leq 0$ for all $i\in S\}$, for $S\neq\{1,2\}$, $S\neq\{3,4\}$ and $S\neq N$.

Individual rationality plus feasibility of x imply that $x_1+x_2+x_3+x_4\leq 1$, $0\leq x_i$. The outcome $x=(0,0,1/2,1/2)$ is simple and it is not in the core, as in the previous example. That x is a PS outcome follows from the fact that $x_1+x_2+x_3+x_4=1$, so x cannot be extended to another simple outcome. Then the game is not singular.

In this example $(1/2,0,0,1/2)$ is in the core. The game is not balanced, since $y\notin VN)$, $y\in V(12)\cap V(34)$ and the collection $\{\{1,2\},\{3,4\}\}$ is balanced. ■

The following example, due to Billera (1970), illustrates Theorem 2, which asserts that: **if the core is non-empty and** $V(N)$ **has a supremum then the game is singular**. In this example the game is not balanced.

Example 4. (**A singular and non-balanced game with non-empty core**). Consider $N=\{1,2,3\}$; $V(1,2,3)=\{x=(x_1,x_2,x_3)\in R^3;\ x_1\leq 0.5,\ x_2\leq 0.5.\ x_3\leq 0\}$, $V(12)=\{x=(x_1,x_2,x_3)\ \in R^3;\ x_1+x_2\leq 1\}$, $V(S)=\{x=(x_1,x_2,x_3)\ \in R^3; x_1\leq 0$ *for all* $i\in S\}$, for all other $S\subseteq N$. This game has a non-empty core consisting of the point (0.5, 0.5, 0), but it is not balanced (the vector y=(1,0,0) is not contained in $V(N)$, even though $y\in V(12)\cap V(3)$ and the collection {1,2},{3} is balanced).

The outcome (0,0,0) is not in the core because it is blocked by $\{1,2\}$. It is a matter of verification that x is simple via $T(x)=\{3\}$ and $S(x)=\{1,2\}$. We claim that $(0,0,0)$ is the only simple outcome out of the core. In fact, let x be a simple outcome, which is not in the core. Let $T(x)$ be a stable coalition for x. Set $S(x)\equiv N-T(x)$. Individual rationality implies that x cannot be blocked by a single individual, so $S(x)\neq\{i\}$ for all $i=1,2,3$, and so $|S(x)|>1$. This implies that $|T(x)|\leq 1$. In any case we have $x=(0,0,0)$.

In this game the only simple payoff vectors are $x=(0,0,0)$ and $y=(0.5,0.5,0)$. Outcome x can be extended to another simple outcome (y is the only PS outcome), so this game is singular. The game is singular and has a non-empty core, although it is not balanced. ■

In the example below $V(N)$ does not have a supremum.

Example 5. (A singular and balanced game with non-empty core). Consider $N=\{1,2,3,4\}$; $V(N)=\{ x=(x_1,x_2,x_3,x_4)\in R^4;\ \sum_{i\in N} x_i\leq 1\}$, $V(12)=\{ x=(x_1,x_2,x_3,x_4)\in R^4;\ x_1+x_2\leq 1/2\}$, $V(34)=\{ x=(x_1,x_2,x_3,x_4)\in R^4;\ x_3+x_4\leq 1/2\}$, $V(S)=\{ x=(x_1,x_2,x_3,x_4)\in R^4;\ \sum_{i\in S} x_i\leq 0\}$, for $S\neq\{1,2\}$, $S\neq\{3,4\}$ and $S\neq N$.

This game has a non-empty core, since $y=(1/4,1/4,1/4,1/4)$ is not blocked by any coalition. We claim that this game is singular. In fact, the outcome $(0,0,0,0)$ is in $V(S)$ for all coalitions S, so it is simple. Clearly, this outcome is not in the core and can be extended to the simple outcome y. Now, let $x\neq(0,0,0,0)$. Suppose x is simple and out of the core. Let $T(x)$ be a stable coalition for x. Set $S(x)=N-T(x)$. Then, $T(x)\neq N$, so $S(x)\neq\phi$. Since x is individually rational then $S(x)\neq \{i\}$ for all $i=1,\ldots,4$. Therefore, $|T(x)|=2$. The possibilities for $T(x)$ are:

a) $T(x)=\{1,2\}$. Then, $x\in V(12)$, so $x_1+x_2=1/2$. Also, $x_i\geq 0$ for $i=1,2$. This outcome can be extended to $z=(x_1,x_2,1/4,1/4)$. It is a matter of verification that z is in the core, so it is simple.

b) $T(x)=\{3,4\}$. In this case, $x\in V(34)$, so $x_3+x_4=1/2$. Also, $x_i\geq 0$, $i=3,4$. This outcome can be extended to $w=(1/4,1/4,x_3,x_4)$. It is easy to see that w is in the core, so it is simple.

Therefore, every simple outcome out of the core can be extended to another simple outcome. Hence, the game is singular. ■

The game of Example 5 is a translation of the TU game v where $v(N)=1$, $v(12)=v(34)=1/2$, $v(S)=0$ for all $S\neq\{1,2\}$, $S\neq\{3,4\}$ and $S\neq N$. Theorem 3 asserts that **every such a game, with non-empty core, is singular.**

4. THE NON-EMPTINESS OF THE CORE

Theorem 1 provides a sufficient condition for the non-emptiness of the core. The idea is that if x is simple and it is not in the core, then x can be extended to a simple outcome x^1. If x^1 is not in the core then it can be extended to a simple outcome x^2, and so on. As proved in Theorem 1, the set of simple outcomes is compact. Then, this sequence converges to a simple outcome, which cannot be extended to another outcome, so it is in the core.

Theorem 1. *Every singular game has non-empty core.*

Proof. Consider a singular game (N,V). We have that the set of simple payoff vectors of (N,V) is non-empty, since the game is singular. Therefore, it is sufficient to show that the set of simple payoff vectors of (N,V) is a compact subset of R^n. In fact, if this is established, then there is a simple payoff vector x^* such that $\sum_{i\in N} x^*_i \geq \sum_{i\in N} x_i$ for all simple outcomes x, so x^* does not have any simple extension. Hence, the core of (N,V) is non-empty because x^* must be in the core.

Then, first observe that the set of simple payoff vectors is bounded, for it is contained in $V(N)\cap R^n_+$. To see that this set is closed, take any sequence $(x^t)_t$ of simple payoff vectors, with $x^t \to x$, as t goes to infinity. We are going to show that x is simple. Without loss of generality we can suppose $x^t \neq (0,\dots,0)$ for infinitely many t's. Since the set of players is finite, there are sets T and S and a subsequence of $(x^t)_t$, such that x^t is simple via T for every term x^t of the subsequence and $S=N-T$. We will use the same notation $(x^t)_t$ for such a subsequence.

We claim that T is a stable coalition for x. In fact, if $S'\subseteq T$, the fact that every term x^t is simple implies that no x^t belongs to $Int V(S')$, so $x \notin Int V(S')$, and so no sub coalition of T can block x. Now, if R blocks x and $R\cap T\neq\phi$, then $R-T$ blocks x. In fact, since $Int\ V(R)$ is an open subset of R^n, if $x\in Int\ V(R)$ then there is some term of the subsequence, say x^t, such that $x^t \in Int\ V(R)$, so R blocks x^t. Since x^t is simple it follows that $x^t \in Int\ V(R-T)$. But $x^t_i=0=x_i$ for all $i\in R-T$. Then Remark 1-b) implies that $x\in Int\ V(R-T)$, and so $R-T$ blocks x. Finally, if $i\in S$ then i is part of some blocking coalition of x. In fact, take $S'\subseteq S$ such that $i\in S'$ and S' is a blocking coalition of x^t for some t (such coalition S' exists because x^t is simple). Then, $x^t\in Int\ V(S')$, so $x\in Int\ V(S')$ by Remark 1-b). Therefore, T is a stable coalition for x. Now use that $V(N)\cap R^n_+$ and $V(T)$ are closed to get that $x\in V(N)\cap R^n_+$ and $x\in V(T)$. Clearly, if $i\in S$ then $x_i=0$. Hence, x is simple via T and the set of simple payoff vectors is closed and bounded, so it is compact. ■

In example 1 the core is empty and the game is not singular.

Scarf (1967) proved that every balanced game has a non-empty core. However, **balancedness of the game is not necessary for non-emptiness of the core,** as shown in examples 2 and 4. In Example 2, $V(N)$ has a supremum given by the vector $(0,0,1/2,1/2)$. The game is non-singular and the core is empty. In Example 4, $V(N)$ has a supremum given by the core vector (0.5, 0.5, 0). The core is non-empty and the game is singular. In both examples the game is not balanced. Theorem 2 asserts that when $V(N)$ has a supremum, **the singularity of a game is a necessary condition for the non-emptiness of the core.**

Theorem 2. *Suppose there is some vector* $M\in R^n$ *such that* $V(N)=\{x\in R^n;\ x_i \leq M_i$ *for all* $i\in N\}$. *If the core of the coalition game* (N,V) *is non-empty then* (N,V) *is a singular game.*

Proof. Suppose the core is non-empty. Then take y in the core. Let x be a simple payoff vector via $T(x)$. Now construct z such that $z_i=x_i$ for all $i\in T(x)$ and $z_i=y_i$ for all $i\in S(x)$. Set $T\equiv T(x)$ and $S\equiv S(x)$. We claim that z is in the core. In fact, if there is some coalition S' which blocks z then there is some $w\in V(S')$ with $w_i>z_i$ for all $i\in S'$. In this case, S' cannot be contained in $T(x)$, since no subset of $T(x)$ blocks x. Also, S' cannot be contained in $S(x)$, otherwise S' would block y, which contradicts the assumption that y is in the core. Therefore, $S'=R\cup R'$, with $R\subseteq T(x)$ and $R'\subseteq S(x)$. But then, $w_i>z_i=x_i$ for all $i\in R;\ w_i>z_i=y_i\geq 0=x_i$ for all $i\in R'$, so S' blocks x via w. But $R\neq\phi$, $R\subseteq T(x)$, so R' blocks x via some w', with $w'_i\geq w_i$ for all $i\in R'$, so $w'\in V(R')$. However, $w'_i>y_i$ for all $i\in R'$, so R' blocks y via w', contradiction. Then z is not blocked by any coalition. The feasibility of z follows from the fact that x and y are in $V(N)$, so, for all $i\in N$, $M_i\geq x_i$ and $M_i\geq y_i$ and so $M_i\geq z_i$. But then, $z\in V(N)$ by (A.3). Hence z is in the core.

It remains to show that z extends x. By construction of z we have that $z_i\geq x_i$ for all $i\in N$. On the other hand, since x is not in the core, x must be different from z. This means that there is some $i\in N$ such that $z_i>x_i$. Hence z extends x, so the game (N,V) is simple and the proof is complete. ■

A transferable utility game v can be translated into a non-transferable utility game V by setting

$$V(S)\equiv\{x\in R^n;\ \sum_{i\in S} x_i\leq v(S)\} \qquad (A.7)$$

for all non-empty coalitions S. When $V(N)$ does not have a supremum but all $V(S)$ are given by (A.7), singularity is still a necessary condition for the non-emptiness of the core, as it is illustrated in Example 5. The following theorem is proved in Sotomayor (2005-c). For the sake of completeness we sketch its proof here.

Theorem 3. *Let* $V(S)\equiv\{x\in R^n;\ \sum_{i\in S} x_i\leq v(S)\}$ *for all non-empty coalitions* S*. If the core of the coalition game* (N,V) *is non-empty then* (N,V) *is a singular game.*

Sketch of the proof. It follows the proof of Theorem 2. The only difference is in the proof of the feasibility of z. This follows from the fact that $v(N)\leq\sum_{i\in N} z_i=\sum_{i\in T} x_i+\sum_{i\in S} y_i=v(T)+\sum_{i\in S} y_i\leq\sum_{i\in T} y_i+\sum_{i\in S} y_i=\sum_{i\in N} y_i=v(N),$ where in the first inequality we used that N does not block z in the second equality we used that x is simple, so $x(T)=v(T)$ and in the last inequality we used that y is in the core. Then, $v(N)=\sum_{i\in N} z_i$, so z is in $V(N)$.■

Theorems 2 and 3 give a criterion to conclude that the core is empty without needing to make many computations. It is enough to find a simple outcome out of the core that cannot be extended to another simple outcome. Example 2 illustrates well this fact when $V(N)$ has a supremum.

5. AN ECONOMIC APPLICATION

The Housing market of Shapley and Scarf (1974) is an example of a well-behaved exchange economy. The set of traders is $N=\{1,2,\ldots,n\}$. Every player $i\in N$ is characterized by means of an initial endowment vector $e^{(i)}$, where $e^{(i)},\ldots,e^{(n)}$ are the unit-vectors in R^n, and a preference ordering $\geq_i$, complete and transitive, defined on $\Omega\equiv\{e^{(1)},\ldots,e^{(n)}\}$. Let u_i be a utility function representing $\geq_i$. We will normalize so that $u_i(e^{(i)})=0$ for all $i\in N$. An *allocation* is a permutation of the unit-vectors $e^{(1)},\ldots,e^{(n)}$. The coalition S blocks the allocation y if there is an allocation $z^{(1)},\ldots,z^{(n)}$, with $\sum_{i\in S} z^{(i)}=\sum_{i\in S} e^{(i)}$ and such that $z^{(i)}>_i y^{(i)}$.

In this section we will define a *non-transferable utility game* V (also called *market game* V) whose core coincides with the core of the Housing market and then, by using Theorem 1, we will prove that the core of this economy is non-empty.

For every non-empty coalition S define:

$V(S)=\{x \in R^n; \exists\, y^{(i)} \in \Omega \;\; \forall\, i \in S,$ *such that* $y^{(i)}=e^{(j)}$ *for some* $j \in S,\ \sum_{i \in S} y^{(i)}= \sum_{i \in S} e^{(i)}$ *and* $x_i \leq u_i(y^{(i)}) \;\; \forall i \in S\}$.

For $S=\phi$ set ($A.1$). Then, $V(i)=\{x \in R^n; x_i \leq 0\}$ and ($A.1$)-($A.5$) are satisfied.

Clearly, the allocation $y^{(1)},\ldots,y^{(n)}$ is in the core of the exchange economy if and only if the vector $(u_1(y^{(1)}),\ldots, u_n(y^{(n)}))$ is in the core of the market game V.

To see that the core of the market game V is non-empty, suppose that x is a simple payoff vector of R^n and that it is not in the core. Let T be a stable coalition for x. Set $S \equiv N\text{-}T$. Since x is not in the core, $S \neq \phi$. By definition of T we have that $x_i=0$ for all $i \in S$ and $x \in V(T)$. We are going to show that x can be extended to a simple payoff vector. In fact, for every $i \in S$ set $u_i(max) \equiv max\{u_i(e^{(t)}),\ t \in S\}$. Now, choose any $i \in S$. Then, there is some other player in S, say j, such that $u_i(max)=u_i(e^{(j)})$ ($i \neq j$ because i is part of a blocking coalition contained in S). Since $j \in S$, there is some agent in S, other than j, say k, such that $u_j(max)=u_j(e^{(k)})$ and so on. Since S is finite, this sequence will cycle. This cycle is a blocking coalition of x. Call it S' Now, define y such that, if $i \in S'$ then $y_i=u_i(max)$; if i is not in S' then $y_i=x_i$. We claim that $T \cup S'$ is a stable coalition for y. In fact, if there is some coalition R which blocks y via some w it also blocks x via w. Then, R cannot be contained in T, since x is simple; also, R cannot be contained in S', because every player i in S' is getting $u_i(max)$ under y. Suppose $R=A \cup B$, with $A \subseteq T$ and $B \subseteq S'$. Then, R also blocks x via w and so B blocks x via some w', so $w' \in V(B)$. Also, $w'_i \geq w_i$ for all $i \in B$. However, $w'_i \geq w_i > y_i$ for all $i \in B$, so B blocks y via w', which contradicts the fact that S' does not contain any blocking coalition of y.

It is clear that $y \in V(T \cup S')$. On the other hand, the fact that x is simple implies that if $i \notin T \cup S'$ then i is part of some blocking coalition of y. Hence, $T \cup S'$ is a stable coalition for y and y is simple. In addition, y extends x. Theorem 1 then implies that the core of this game is non-empty, consequently, the core of the exchange economy of Shapley and Scarf is also non-empty.

6. CONCLUDING REMARKS AND RELATED WORK

In this paper we approach a coalitional game with non-transferable payoff by focusing on simple outcomes, aiming to deal with the existence problem of core outcomes. A new point of view is provided through the concepts of simple outcome and Pareto optimal simple outcome. Core outcomes are simple and Pareto optimal, but not all Pareto optimal outcomes are in the core or are Pareto optimal simple. Outcomes out of the core may be Pareto optimal and simple outcomes out of the core may be Pareto optimal simple. We first proved **that if no simple outcome out of the core is Pareto optimal simple then the core is non-empty.** This is equivalent to say that a singular game has non-empty core. Different sufficient conditions, based on the balancedness and on π-balancedness of the game, have also been obtained by Scarf (1967), Billera (1970) and Shapley (1973). However, these conditions are not necessary.

A necessary and sufficient condition for the non-emptiness of the core was identified by Billera (1970), under the assumption that the sets $V(S)$ are all convex. Then, this condition does not apply to a general NTU game.

We did not assume convexity for all $V(S)$. It is only required that there is some vector $M \in V(N)$, with $M_i \geq 0$ for all $i \in N$ and such that $x_i \leq M_i$ for all $x \in V(N)$. For this class of games we proved that the singularity condition is also necessary for the non-emptiness of the core: *When the core is non-empty, every Pareto optimal simple outcome is in the core.* If $V(N)$ does not have a supremum, Example 3 shows that we may have non-empty cores and a Pareto optimal simple outcome out of the core. However, although the set $V(N)$ does not have a supremum in the general games with transferable payoff, the singularity condition is also necessary for the non-emptiness of the core of such games.(Sotomayor (2005-c))

This approach has been used for several models, where the concept of simple outcome has been slightly modified. Its main feature is to prove intuitive and important results without the framework of sophisticated mathematical tools.

Recently, natural adaptations of the concept of simple outcome were introduced in Sotomayor (2005-a), (2005-b) and (2005-d). The first paper proves that the core of the Roommate model of Gale and Shapley (1962) is non-empty if and only if no simple outcome out of the core is Pareto optimal simple. In the second paper, this same result is proved for a TU version of the roommate-model, the one-sided Assignment game. In the latter, this condition is proved to be always satisfied for the Housing market with strict preferences of Shapley and Scarf (1974); so the core is always non-empty in this market. In all these models the set of simple outcomes is non-empty and so a PS exists.

As for two-sided matching markets, a version of the concept of simple outcome has been introduced in: (a) Sotomayor (1996), for the Marriage market, (b) Sotomayor (1999), for the College Admission model and the discrete many-to-many matching model with substitutable and non-strict preferences and in (c) Sotomayor (2000), for the two-sided Assignment game of Shapley and Shubik (1972) and the unified two-sided matching model of Eriksson and Karlander (2000). In each of these models, a non-constructive existence proof of pairwise-stable outcomes has been provided by showing that simple outcomes that are pairwise unstable are not Pareto optimal simple.

We believe that the theory developed here, the concepts of simple outcome and Pareto optimal simple outcome, open a new way to study coalitional games. Since the theory for the case with infinitely many players springs from the study of games with finite number of players, we conjecture that the insights gained with this new approach can be useful to these more general games. This conjecture is a challenge that we intend to investigate in the future.

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