Department of Economics- FEA/USP

On the Core of the Coalitional Games with Transferable Payoff and Finite Set of Players

MARILDA SOTOMAYOR

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Marilda Sotomayor (marildas@usp.br)

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A dynamic game where agents "behave cooperatively" is postulated: At each stage, current non-trading agents can trade and the payoffs from transactions done are maintained in the subsequent stages. The game ends when no interaction is able to benefit the agents involved (case in which a core outcome is reached) or when any new interaction requires that some of the agents involved do not "behave cooperatively" (case in which the core is empty).
The gains in terms of insights, obtained with this approach, allow us to identify a new condition, which is proved to be necessary and sufficient for the non-emptiness of the core. The proof of this result is elementary, in the sense that it avoids specialized mathematical tools, and only uses simple combinatorial arguments.

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by

MARILDA SOTOMAYOR[1]

Department of Economics

Universidade de São Paulo, Cidade Universitária, Av. Prof. Luciano Gualberto 908

05508-900, São Paulo, SP, Brazil

e-mail: marildas@usp.br

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ABSTRACT

A dynamic game where agents "behave cooperatively" is postulated: At each stage, current non-trading agents can trade and the payoffs from transactions done are maintained in the subsequent stages. The game ends when no interaction is able to benefit the agents involved (case in which a core outcome is reached) or when any new interaction requires that some of the agents involved do not "behave cooperatively" (case in which the core is empty).

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INTRODUCTION

The traditional cooperative theory of von Neuman and Morgenstern postulates a pre-game where players are permitted to freely communicate, make binding agreements and preferences over outcomes, as well as the rules of the game are common knowledge. If agents "behave cooperatively", and the core is non-empty, then one can expect to observe a core outcome. The premise of cooperative behavior has the natural meaning: *A group of agents will only emerge with an agreement if its members believe that more favorable terms cannot be obtained elsewhere.*

However, the history underlying a core outcome, as well as whether all agreements are made at once or sequentially, make no difference at all. It is a matter for the players to work out for themselves in order to get a specific agreement.

Of course, every economic model involves some compromises with reality, and the cooperative game approach, as a static game, is probably an acceptable compromise for the study of the properties of the core. But being the core the solution concept of games where the agents' principal activity is the formation of coalitions, scenery where the coalitions are not necessarily formed at once seems to be what in reality we have.

With this in mind, we focus on the pre-game reformulated as a finite "dynamic game", whose final outcome is a core outcome, when it exists. At each stage, if it is possible for some group (blocking coalition) of current non-trading agents to "interact", that is, to agree about their payoffs under the premise of cooperative behavior, then an outcome results. The intuitive idea underlying such outcomes is captured by the concept of *simple outcome*, to be defined in the text. For a coalitional game with transferable payoff, (N,v), every simple outcome is feasible and individually rational, and every blocking coalition, if any, has a sub-blocking coalition formed with agents whose payoffs are their reservation payoffs. In games in which, for every coalition S, $v(S)$ is greater than or equal to the sum of the reservation payoffs of all agents in S, the vector of reservation payoffs is a simple outcome. Therefore, the set of simple outcomes is non-empty for these games. Clearly, the core is a subset of the set of simple outcomes.

The present study represents part of a continuing investigation of the properties of the *simple outcomes* aiming to deal with the existence problem of core outcomes. For this purpose, the present paper shows that the gains in insights obtained with our new approach make it more attractive than the traditional one. Indeed, it allows a necessary and sufficient condition for the non-existence of the core to be naturally identified.

To illustrate this point, suppose that players are playing the dynamic game. The cooperative behavior implies that the trading agents' payoffs obtained at a given stage are maintained in the subsequent stages. In this sense, the outcome produced at a given stage "extends" the resulting outcomes from the previous stages. The payoff vectors that we can expect to observe at any stage of the game are, therefore, simple outcomes. It is then intuitive that the game ends when no interaction is able to benefit the agents involved, or when any new interaction requires that some of the agents involved do not behave cooperatively. In the first case, a particular core outcome is reached. In the other case the core is empty.

This intuition is confirmed here. By defining a game to be *singular* if the set of simple outcomes is non-empty and any simple outcome out of the core can be extended to another simple outcome, we can state our main result as follows:

The core of the game (N,v) is non-empty if and only if the game is singular.

If the game is singular, the core is non-empty because no simple outcome out of the core is Pareto optimal among all simple outcomes, and such a Pareto optimal simple outcome always exists, since, in this case, the set of simple payoffs is non-empty and compact.

An example with three players, such that the payoff of any coalition with at least two players is 1, and the reservation payoff of any player is zero, illustrates a game that has an empty core. In this example the outcome where every one gets zero is the only simple outcome of the game. Thus, if some players decide to interact in order to get a higher payoff, the resulting outcome will not be simple, so at least one of the players is not behaving cooperatively.

In economic practical terms, simple outcomes provide an economic intuition of how non-trading agents can do blocking when the core is non-empty. Of course, a simple outcome out of the core is blocked by at least one pair of non-trading agents. If the core is non-empty, this blocking can be done in such a way that the payoffs from the trades done do not change if players reach the core. Moreover, as it is proved here, *the total sum of these payoffs is the same at any core outcome.*

Unlike Bondareva (1963) and Shapley (1967), our proof is not based on balancedness of games. It is elementary, in the sense that it avoids specialized mathematical tools, and only uses simple combinatorial arguments.

This paper is organized as follows. Section 2 describes the game and presents the conceptual framework to be used in section 3. Section 3 presents the results. Section 4 concludes the paper and discusses related work.

2. DESCRIPTION OF THE GAME AND CONCEPTUAL FRAMEWORK

There is a finite set of players N, with $|N|$ players. Associated with any subset (coalition) S of N there is a real number $v(S)$, the worth of the coalition S. Formally,

Definition 1. *A **coalitional game with transferable payoff** consists of (i) a finite set N of players; (ii) a function v that associates with every non-empty subset S of N a real number $v(S)$.*

An outcome for this game is a vector $x \in R^{|N|}$ called payoff vector. The intuitive meaning is that the ith player "receives" x_i. We let $x(S)=\sum_{i \in S} x_i$ denote the total payoff of coalition S. The payoff vector x is **feasible** if $x(N) \leq v(N)$. It is **individually rational** if $x(i) \geq v(i)$ for all $i \in N$.

Definition 2. *The payoff vector x is in the **core** of (N,v) if x is feasible and $x(S) \geq v(S)$ for all $S \subseteq N$.*

If $x(s)<v(S)$, for some S, we say that S *blocks* x. A definition that is obviously equivalent is: *The feasible outcome x is in the core of (N,v), if there is no coalition S and feasible outcome y, for which $y_i > x_i$ for all $i \in S$ and $y(S) \leq v(S)$.* This is to say that x is in the core if it is not blocked by a coalition S via a feasible outcome y.

Given a feasible payoff vector x let $T(x)$ be such that no subset of $T(x)$ can block x. If $i \notin T(x)$ then i is part of some blocking coalition of x. Moreover, if S blocks x via y and $S \cap T(x) \neq \phi$ then $S\text{-}T(x)$ blocks x via y', where $y'_i \geq y_i$ for all $i \in S\text{-}T(x)$. The idea is that the agents in $S\text{-}T(x)$ can do better trades among them rather than with the agents of $T(x)$. Set $S(x) \equiv N\text{-}T(x)$. Of course, $T(x)$ may be empty, in which case $S(x)=N$.

Definition 3. *The payoff vector* x *is* ***simple*** *if (i) it is feasible, (ii) individually rational, (iii)* $x_i=v(i)$ *for all* $i \in S(x)$ *and (iv)* $\sum_{i \in T(x)} x_i = v(T(x))$.

Clearly, any core payoff vector is simple. In this case $T(x)=N$. In some games it is natural to assume that every coalition S can undertake some joint activity that produces the gain $v(S)$ and it is feasible for S to be formed, that is

$$\boldsymbol{v(S) \geq \sum_{i \in S} v(i) \text{ for all } S \subseteq N.}$$

In this case, the set of simple payoff vectors is non-empty, since the payoff vector x where every player i receives $v(i)$ is simple: x is feasible, individually rational and if $S(x) \neq \phi$ and $i \in S(x)$ then $x_i=v(i)$. Since $T(x)$ does not block x we have that $\sum_{i \in T(x)} v(i) = \sum_{i \in T(x)} x_i \geq v(T(x))$. The assumption on v implies that $v(T(x)) \geq \sum_{i \in T(x)} v(i)$. Thus, $\sum_{i \in T(x)} x_i = v(T(x))$ and x is simple.

We must point out that the set $T(x)$ is the main ingredient of a simple outcome. The idea behind a simple outcome x is that $T(x)$ is formed when agents act cooperatively and succeed in their interactions. If $T(x)=\phi$ then no cooperative interaction exists. The dynamic game described in Introduction reflects the properties of $T(x)$, but it does not provide an algorithm to find this set. If we have an algorithm to find $T(x)$, then we have a procedure to find a simple outcome and, consequently, a core outcome when it exists.

Definition 4. *The simple payoff vector* x^* ***extends*** *the simple payoff vector* x *if* $x^*_i \geq x_i$ *for all* $i \in N$ *with strict inequality for at least one* $i \in N$.

Definition 5. *The payoff vector* x *is* ***Pareto optimal simple*** *(PS for short) if it is simple and there is no simple payoff vector that extends* x.

Therefore, if x is PS and $y_i > x_i$ for some player i and some simple payoff vector y, then there exists some other player k such that $y_k < x_k$.

Definition 6. *The game* (N,v) *is* ***singular*** *if (i) the set of simple outcomes is non-empty and (ii) every simple outcome out of the core, if any, has a simple extension.*

3. RESULTS

The following example shows that the core of (N,v) may be empty.

Example 1. Consider $N=\{1,2,3\}$, $v(i)=0$ for all $i=1,2,3$ and $v(S)=1$ for all $S\subseteq N$ such that $|S|\geq 2$. Given any payoff vector x, there will exist two players i and j such $x_i + x_j<1=v(i,j)$. Hence x is not in the core. ∎

Proposition 1 asserts that the set of simple payoff vectors is compact. Then, when this set is non-empty the existence of a Pareto optimal simple outcome is guaranteed.

Proposition 1. *The set of simple payoff vectors is a compact set of* $R^{|N|}$.

Proof. The feasibility and individual rationality of a simple outcome x implies that $v(i)\leq x_i\leq v(N)$ for all $i\in N$. Thus the set of simple outcomes is bounded. To see that this set is closed, take any sequence $(x^t)_t$ of simple payoff vectors, with $x^t\to x$, as t tends to infinity. Since the set of players is finite, there is some set T and some set S such that $T=T(x^t)$ and $S=S(x^t)$ for infinitely many t's. We will use the same notation $(x^t)_t$ for this subsequence. Then, $x(T)=\lim_{t\to\infty} x^t(T)=v(T)$. Also, if $i\in S$ then $x_i=\lim_{t\to\infty} x^t_i=\lim_{t\to\infty} v(i)=v(i)$. It is also easy to see that x is feasible and individually rational. Therefore, x is simple. Hence, the set of simple payoff vectors is closed and bounded, so it is compact.∎

Our main result is Theorem 2 below. It provides a necessary and sufficient condition for the non-emptiness of the core. If the condition is satisfied then PS outcomes exist and any PS is in the core. If the core is non-empty, then every PS is in the core. Thus, an equivalent statement of Theorem 2 is: *The core is non-empty if and only if simple outcomes exist and no simple outcome out of the core is Pareto optimal simple.*

Theorem 2. *The core of the coalitional game* (N,v) *is non-empty if and only if* (N,v) *is singular.*

Proof. Suppose the core is non-empty. Then the set of simple outcomes is non-empty. If there is no simple outcome out of the core then (N,v) is singular. Otherwise, let x be a simple payoff vector out of the core. Take y in the core. Now construct z such that $z_i=x_i$ for all $i\in T(x)$ and $z_i=y_i$ for all $i\in S(x)$. Set $T\equiv T(x)$ and $S\equiv S(x)$. We claim that z is in the core. In fact, if there is some coalition S' which blocks z via a feasible outcome w, then S' cannot be contained in T, for if so S' would block x, contradicting the definition of T. Also, S' cannot be contained in S, for if so S' would block y, which contradicts the assumption that y is in the core. Then, $S'=R\cup R'$, with $R\subseteq T$ and $R'\subseteq S$. But then, $w_i>z_i=x_i$ for all $i\in R$ $w_i>z_i=y_i\geq v(i)=x_i$ for all $i\in R'$, so S' blocks x via w. But $R\neq\phi$ and $R\subseteq T(x)$. By definition of $T(x)$ we must have that R' blocks x via some w', with $w'_i\geq w_i$ for all $i\in R'$, so $\sum_{i\in R'} w'_i \leq v(R')$. But $w'_i>y_i$ for all $i\in R'$, so R' blocks y via w', contradiction. Then z is not blocked by any coalition. The feasibility of z follows from the fact that $v(N)\leq\sum_{i\in N} z_i=\sum_{i\in T} x_i + \sum_{i\in S} y_i=v(T) +\sum_{i\in S} y_i \leq \sum_{i\in T} y_i+\sum_{i\in S} y_i=\sum_{i\in N} y_i=v(N)$, where in the first inequality we used that N does not block z (recall that we just proved that z does not have a blocking coalition), in the second equality we used that x is simple, so $x(T)=v(T)$, and in the last inequality we used that y is in the core. Then, $v(N)=\sum_{i\in N} z_i$. Hence z is in the core, so it is simple.

Now, we are going to show that z extends x. By construction of z we have that $z_i\geq x_i$ for all $i\in N$. On the other hand, since x is not in the core, x must be different from z. This means that there is some $i\in N$ such that $z_i>x_i$. Hence z extends x, so (N,v) is singular.

In the other direction, suppose (N,v) is singular. Then the set of simple outcomes is $\neq\phi$. Define a simple outcome x^* such that $\sum_{i\in N} x^*_i\geq\sum_{i\in N} x_i$, for all simple outcome x. It follows from Proposition 1 that x^* is well defined. Outcome x^* is clearly PS. If x^* is not in the core then, since (N,v) is singular, there exists some simple payoff vector, say z', such that z' extends x^*, contradicting the fact that x^* is PS. Hence, x^* is in the core.∎

In Example 1 above, $x=(0,...,0)$ is simple and is not in the core. However it is the only simple outcome, so it does not have a simple extension and so the core is empty, implied by Theorem 2.

Proposition 3 reinforces the idea of "stability" of the set of trades done under a simple outcome *x,* which is reflected by the conclusion of Theorem 2: Not only there is a core outcome that keeps all trades done under *x,* but also the sum of the payoffs of the trading agents is the same under every core outcome.

Proposition 3. *Let x be a simple payoff vector and let y be in the core. Then, $x(T(x))=y(T(x))$.*

Proof. If x is in the core then $T(x)=N$ and we are done. Then suppose x is not in the core. Define the payoff vector z as in the proof of Theorem 2. Then, $v(N)=\sum_{i\in N} z_i = \sum_{i\in T(x)} x_i + \sum_{i\in S(x)} y_i = v(T(x)) + \sum_{i\in S(x)} y_i \leq \sum_{i\in T(x)} y_i + \sum_{i\in S(x)} y_i = \sum_{i\in N} y_i = v(N)$, so $v(T)=\sum_{i\in T(x)} y_i$. Hence, $\sum_{i\in T(x)} x_i = \sum_{i\in T(x)} y_i$, and the proof is complete.∎

4. CONCLUDING REMARKS AND RELATED WORK

The aim of this paper was to deal with the existence problem of core outcomes for a general coalitional game with transferable payoff. This problem has already been treated by Bondareva (1963) and Shapley (1967), by making use of the Theory of balanced games. A new point of view was provided here through the concepts of simple outcome and Pareto optimal simple outcome. Core outcomes are simple and Pareto optimal, but not all Pareto optimal outcomes are in the core or are Pareto optimal simple. Outcomes out of the core may be Pareto optimal and simple outcomes out of the core may be Pareto optimal simple. However, we proved that the core is non-empty if and only if simple outcomes exist and no simple outcome out of the core is Pareto optimal simple. Hence, when the core is non-empty, every Pareto optimal simple outcome is in the core.

This approach have been used for several models, where the concept of simple outcome has been slightly modified. Its main feature has been to provide insight to prove intuitive and important results without the framework of sophisticated mathematical tools.

Recently, natural adaptations of the concept of simple outcome were introduced in Sotomayor (2005-a), (2005-b) and (2005-c). The first paper proves that the core of the Roommate model of Gale and Shapley (1962) is non-empty if and only if no simple outcome out of the core is Pareto optimal simple outcome. In the second paper, this same result is proved for a TU version of the roommate-model, the one-sided Assignment game.

In the latter, this condition is proved to be always satisfied for the Housing market with strict preferences of Shapley and Scarf (1974); so the core is always non-empty in this model. In all these models the set of simple outcomes is non-empty and so a PS exists.

As for two-sided matching markets, a version of the concept of simple outcome has been introduced in: a) Sotomayor (1996), for the Marriage market, b) Sotomayor (1999), for the College Admission model and the discrete many-to-many matching model with substitutable and non-strict preferences and c) Sotomayor (2000) for the two-sided Assignment game of Shapley and Shubik (1972) and the unified two-sided matching model of Eriksson and Karlander (2000). In each of these models a non-constructive existence proof of pairwise-stable outcomes has been provided by showing that simple outcomes that are pairwise unstable are not Pareto optimal simple outcomes. Pareto optimal simple outcomes always exist for these models too.

We believe that the theory developed here, the concepts of simple outcome and Pareto optimal simple outcome, open a new way to study the existence problem of core outcomes of more general coalitional games.

REFERENCES

Bondareva, O. N. (1963)"Some applications of linear programming methods to the theory of cooperative games", *Problemy Kybernetiki,* 10: 119-139 [in Russian]

Eriksson, K. and Karlander, J. (2000) "Stable matching in a common generalization of the marriage and assignment models", *Discrete Mathematics* 217, 135-156.

Gale,D. and Shapley, L. (1962) “College admissions and the stability of marriage”, *American Mathematical Monthly* 69, 9-15.

Shapley, L. S. (1967) "On balanced sets and cores", *Naval Research Logistics Quarterly,* 14, 453-460.

Shapley, L. and Scarf, H. (1974) “On cores and indivisibility” *Journal of Mathematical Economics,* 1, 23-8.

Shapley, L. and Shubik, M. (1972) "The assignment game I: The core", *Journal of Game Theory* 1, 111-130.

Sotomayor, M. (2005-a) "The roommate problem revisited", *mimeo.*

______________(2005-b) "On the core of the one-sided assignment game", *mimeo.*

____________ (2005-c) "An elementary non-constructive proof of the non-emptiness of the core of the housing market of Shapley and Scarf", *Mathematical Social Sciences, to appear.*

___________(1999) "Three remarks on the many-to-many stable matching problem", *Mathematical Social Sciences* 38, 55-70

___________(2000) "Existence of stable outcomes and the lattice property for a unified matching market", *Mathematical social Science* 39 , 119-132.

___________(1996) "A non-constructive elementary proof of the existence of stable marriages", *Games and Economic Behavior* 13, 135-137.