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Detecting Bubbles in the Brazilian Commercial Real Estate Market: 2012-2023

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WORKING PAPER SERIES Nº 2024-08

DEPARTMENT OF ECONOMICS, FEA-USP
WORKING PAPER Nº 2024-08

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This research examines the dynamics of commercial real estate prices in the Brazilian market, exploring the potential presence of speculative movements within the real estate market for the period 2012-2023. The study utilizes a conventional present value asset pricing model with a consistent discount factor and employs established bubble tests from the finance literature. These tests encompass assessments for explosive bubbles, periodic bubbles, multiple explosive bubbles and intrinsic bubbles. As a result, this approach allows us to diagnose unsustainable trends in the trajectory of real estate asset prices, should speculative bubbles be detected within the series. Utilizing data from the Fipezap survey provided by the Institute of Economic Research Foundation (FIPE), the study identifies evidence of periods marked by price exuberance, as indicated by the explosive bubble test and the multiple bubble test for some cities and the national price index. However, using the corresponding tests, no evidence of neither periodically collapsing bubbles nor intrinsic bubbles was found across any of the considered cities

Keywords: Real Estate, Speculative Bubbles, Brazilian Empirical Analyses

JEL Codes: C32, G12

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Abstract

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1 Introduction

The real estate market is extremely important in the Brazilian economy, as it plays a role in driving the construction sector and provides essential goods and services. Therefore, understanding the dynamics of real estate prices is essential for evaluating the allocative efficiency of capital, the growth rate of the sector, and the risk of market crises. This research examines the dynamics of prices in the real estate market and investigate the presence of speculative bubbles, which refers to the deviation of asset prices from their fundamentals. The focus of this research will be on the commercial real estate market in Brazil, covering the period from 2012 to 2023.

In the case of Brazil, investment in the real estate market historically represents a significant portion of investors' portfolios. This is partly due to uncertainties regarding inflation rates, which have been high and volatile during certain periods of the country's history, as well as limited access to financial assets for a large portion of the population. As a result, investing in physical assets, particularly real estate, has been considered a relatively safe option by many Brazilian investors.

*Mira acknowledges the financial support of the Capes 88887.673227/2022-00 and Fapesp 2023/02461-7. Maldonado acknowledges the financial support of the CNPq 302856/2022-6 and 402949/2021-8, and the Fundação Instituto de Pesquisas Econômicas - FIPE. Enrico Campos de Mira e-mail: enrico.c.mira@usp.br. Wilfredo Leiva Maldonado e-mail: wilfredo.maldonado@usp.br

To better understand this market, the research project aims to conduct tests to determine the presence of bubbles in the Brazilian real estate market. We will delve into investigating the presence of speculative bubbles, as highlighted by Stiglitz (1990). A bubble is said to occur when the price of an asset increases due to the anticipation that it can be sold in the future at a higher price, despite lacking indications in the fundamentals to support this. In other words, it's an upward price movement caused by a self-fulfilling expectation among agents. In this regard, our objective is to identify, based on the results obtained from our pricing model, the portion of price series movements in assets that aligns with the dynamics of the modeled fundamentals, and the portion that possibly reflects the presence of speculation concerning property prices (M. W. Blanchard W. 1982; R. P. Flood, Hodrick, and Kaplan 1986; West 1987).

Nevertheless, as pointed out by M. W. Blanchard W. (1982), bubbles can assume various forms and are often challenging to diagnose. This challenge arises from two primary factors. Firstly, distinguishing between the effect of a hypothetical asset price bubble and the impact of omitted variables that influence the fundamentals is intricate. A common example pertains to agent expectations, which, even though not directly observable, can lead a researcher to mistakenly attribute a bubble to variations not explained by the fundamental model. The second factor pertains to the potential misspecification of the fundamental itself (R. Flood and Garber 1980; Hamilton and Whiteman 1985). Thus, should we detect price variations unexplained by the fundamentals, we cannot assert the presence of a bubble without first substantiating the validity of the considered fundamentals.

In light of this, there has been a noteworthy surge in recent research dedicated to detecting and analyzing rational bubbles. Exploring various facets of the theory of rational bubbles, which forms the foundational framework for multiple bubble testing methodologies. Articles like the one written by Maldonado and Ribeiro (2017) and Maldonado, Ribeiro, and Tourinho (2021) have employed several different bubble tests, primarily aimed at identifying potential bubbles and have also investigated the relationships between different types of bubbles.

Consequently, in line with other authors, to diagnose bubbles, we will initially test for the presence of explosive bubbles. As noted by M. W. Blanchard W. (1982), O. J. Blanchard (1979), Lee and P. C. Phillips (2016), E. G. Pavlidis, Paya, and Peel (2017), and P. C. B. Phillips, Shi, and Yu (2015a,b), rational decision-making by agents in a dynamic and uncertain environment can yield asset price trajectories that exhibit explosive behavior. For this purpose, we will utilize the unit root tests presented by B. Diba and H. Grossman (1984) and the cointegration tests between observed prices and estimated fundamental values proposed by J. Campbell and Shiller (1987, 1988) and B. T. Diba and H. I. Grossman (1988) to diagnose such behavior.

Although widely employed, traditional unit root tests exhibit remarkably low efficacy in identifying instances of explosive dynamics when they are interrupted by market crashes (Evans 1991). In order to enhance the previous analysis and further explore the concept of cointegration among the series, we will employ Bohl (2003) formulation to test for the presence of bubbles as proposed by Evans (1991). Evans (1991) propose a periodically bubble model in which the bubble undergoes a three-stage process. In the initial phase, the bubble experiences growth at a constant rate. Once it surpasses a certain threshold value, the rate of expansion intensifies, and within this interval, there is a possibility of a subsequent collapse. The collapse process is governed by a

Bernoulli process. Once the bubble collapses, the process restarts, characterizing it as a periodic process.

In Bohl (2003)'s investigation of the US Standard and Poor's stock price index spanning from 1871 to 2001, the Momentum Threshold Autoregressive (MTAR) test was employed. The study concluded that a periodically collapsing bubble (PCB) was observed during the entire period, 1871 to 2001, but notably, this phenomenon did not persist in the sub-period from 1871 to 1995. Bohl (2003)'s methodology, introduced in this research, has since been widely utilized in empirical studies to identify asymmetric adjustments in the dynamics of asset prices.

J. Payne and Waters (2007) extended the application of the MTAR test and the residual-augmented Dickey–Fuller (RADF) tests to explore the US All, REIT, Mortgage, and Hybrid REIT indices from 1972 to 2005. Their findings indicated a periodically collapsing bubble, particularly evident in the Mortgage REIT sector. In a second article, examining the US Equity REIT (1973–2003) and sub-sector REIT (1991–2003), Waters and J. E. Payne (2007) employed the MTAR and RADF tests. The results revealed that while the MTAR test did not support the presence of a periodically collapsing bubble, the RADF test suggested the contrary for both the Equity REIT and sub-sector REIT indices.

S.-W. Chen, Hsu, and Xie (2016) delved into the analysis of four international stock markets—S&P 500, BEL 20, FTSE Denmark, and FTSE Finland—utilizing the MTAR unit root test and the LNV-MTAR unit root test. The study concluded that there was no evidence supporting the presence of a periodically collapsing bubble in these stock markets. S.-W. Chen, Hsu, and Xie (2016)'s work contributes to the broader understanding of market dynamics across different international contexts.

A third methodology of diagnosis of bubble that we will also employ is the test propose by P. C. B. Phillips, Shi, and Yu (2015a,b). In order to deal with the effect of a collapse in a time series on the test's performance, the authors propose a GSADF methodologies involve a recursively evolving algorithm that estimates ADF regressions on subsamples of data. These methods will not only enable us to assess a single bubble over the series but also examine the possible presence of multiple bubbles throughout. These tests will also allow us to detect and date the periods in the series exhibiting speculative bubbles.

The GSADF procedure is notably appealing due to its ability to minimize the influence of prior boom-bust episodes on current identification, thus remaining consistent with multiple changes in regime. Empirical evidence from simulations conducted by Homm and Breitung (2012), E. G. Pavlidis, Paya, and Peel (2017), and P. C. B. Phillips, Shi, and Yu (2015a) suggests that this test exhibits accurate size and higher power compared to alternative tests for changes in persistence. Another advantageous aspect of the GSADF methodology is its recursive nature, enabling the precise dating of periods during which the examined series displays explosive dynamics.

Consequently, it proves useful not only for shedding light on past episodes of exuberance but also for real-time market surveillance. The GSADF are univariate testing procedures, limiting conclusions to the unit level. However, exuberance often occurs concurrently across a group of assets, such as regional house prices or stock prices. To address simultaneous episodes of exuberance, Efthymios Pavlidis, Yusupova, et al. (2016) propose extending the GSADF procedure to a panel setting. The panel GSADF draws inferences on overall exuberance by leveraging the cross-sectional dimension of a dataset through a sieve bootstrap procedure. This extension can significantly outperform univariate tests applied to aggregated series in the presence of synchronized episodes of exuberance and, like univariate tests, provides a date-stamping strategy (Efthymios Pavlidis, Martínez-García, and V. Grossman 2019; Vasilopoulos, Efthymios Pavlidis, and Martínez-García 2022).

Finally, we will conduct tests for intrinsic bubbles. This concept of intrinsic bubbles, introduced by Froot and Obstfeld (1991), differs from the aforementioned speculative bubbles as they represent deviations of observed prices from fundamental values due to nonlinear variations in the fundamentals. In this case, changes in the fundamentals influence the size of the bubble, and the applied test aims to capture this pattern. Nneji, Brooks, and Ward (2013) employ a similar methodology to test intrinsic bubbles in the US real estate market between 1960 and 2011.

This research will further unfold in five distinct sections. Section 2 outlines the research methodology, presenting the bubble tests to be utilized. Data will be presented in Section 3. Empirical test results for the Brazilian commercial real estate market data are expounded in Section 4. Finally, the project culminates in Section 5, providing a conclusive summary.

2 Research Method

In this section, we provide a brief overview of the Real Estate Asset Pricing and Rational Bubble Definition. Then, we introduce the bubble tests: explosive, periodically, multiple, multiple in panel and intrinsic, which will be used in this research.

2.1 Real Estate Asset Pricing and Rational Bubble Definition

Rational bubbles in real state markets occur when asset prices deviate at a geometric rate from their fundamental value. A standard theory-based approach to defining a rational bubble commences with the accounting identity of real asset returns (R_t) over the period $[t, t+1]$, expressed by the equation:

$$1 + R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} \tag{1}$$

Where P_t is the real price of the real estate asset at the beginning of period t and D_t is the real rent paid at the period $t+1$. Following M. W. Blanchard W. (1982) and J. Y. Campbell, Lo, and MacKinlay (1997), to obtain the fundamental price of the asset we take the conditional expectation in time t of the equation (1) leads to the standard no arbitrage condition:

$$P_t = \beta_t E_t[P_{t+1} + D_{t+1}] \text{ with } \beta_t = \frac{1}{1 + E_t[R_{t+1}]} \tag{2}$$

Where $E_t[.]$ is the expectation operator conditional on informational at the beginning of period t and the discount factor $\beta_t \in (0, 1)$. [1]

[1] Log-linear approximations are frequently employed, although their applicability might diminish when dealing with nonstationary

2.2 Constant Discount

At this point, the classical theory assumes the conditional stationarity of the real asset return, which means that $E_t[R_{t+1}]$ is constant, allowing us to define a constant discount factor $\beta = \beta_t \in (0, 1)$. The equation (2) can be routinely solved by substituting future prices forward repeatedly. So, we obtain a present-value formula for the stock price at time t:

$$P_t = \sum_{k=0}^{\infty} \beta^k E_t[D_{t+k}] + B_t \tag{3}$$

Where P_t^f is the fundamental of the commercial real estate prices that can be define as $P_t^f \equiv \sum_{k=0}^{\infty} \beta^k E_t[D_{t+k}]$ and B_t is the bubble component that can be define as $B_t \equiv \lim\limits_{T\to\infty} \beta^T E_t[P_{t+T}]$. If $d_t = \log(D_t)$ is a random walk process, satisfying $d_t = \mu + d_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim iid(0, \sigma^2)$, then we can rewrite P_t^f as:

$$P_t^f = \sum_{k=0}^{\infty} \beta^k E_t[D_{t+k}] = \sum_{k=0}^{\infty} \beta^k E_t[D_t e^{\mu+\varepsilon_{t+k}}] = \sum_{k=0}^{\infty} \beta D_t e^{(\mu+\frac{\sigma^2}{2}+\log(\beta))k} \tag{4}$$

If the parameters satisfy $\mu + \frac{\sigma^2}{2} + \log(\beta) < 0$, we will have that the series converges to:

$$P_t^f = \frac{\beta}{\left(1 - \beta e^{\left(\mu+\frac{\sigma^2}{2}\right)}\right)} D_t = \frac{1}{\left(\beta^{-1} - e^{\left(\mu+\frac{\sigma^2}{2}\right)}\right)} D_t \tag{5}$$

The B_t is the bubble component that can be define as $B_t \equiv \lim\limits_{T\to\infty} \beta^T E_t[P_{t+T}]$. The bubble component B_t satisfies the (discounted) martingale property, i.e.:

$$\begin{aligned} \beta E_t[B_{t+1}] &= \beta E_t[\lim_{T\to\infty} \beta^T E_{t+1}[P_{t+1+T}]] = \\ &= E_t[\lim_{T\to\infty} \beta^{T+1} E_{t+1}[P_{t+1+T}]] \end{aligned} \tag{6}$$

Define $N = T + 1$:

$$\beta E_t[B_{t+1}] = \lim_{N\to\infty} \beta^N E_t[P_{t+N}] = B_t \tag{7}$$

The presence of B_t in equation (3) is called a rational bubble because it aligns with the principles of rational expectations. Notice that equation (7) implies that $E_t[B_{t+T}] = \beta^{-T} B_t$, thus if $B_t > 0$ in some instant t, then it is expected the increase of B_{t+T} at a geometric rate, so the spot price P_t departs from the fundamental value P_t^f at a geometric rate.

According to equation (5), it is evident that commercial real estate prices P_t can display explosive dynamics even in the absence of a bubble, primarily due to explosive dynamics in fundamentals - rents R_t. In such cases, exuberance in the commercial real estate market is inherited from fundamental factors.

To further explore this analysis, we not only evaluate the real price but also investigate the price-to-rent ratio. Conversely, in the presence of bubbles, prices surge as expectations outpace fundamentals, resulting in

data in which sample averages fail to converge to constant population values, as discussed by J. Campbell and Shiller (1988) and J. Y. Campbell, Lo, and MacKinlay (1997). Additional insights into these approximations can be located in the work by Lee and P. C. Phillips (2016). In our study, we operate at the level of the data, and employing logarithmic transformations does not fundamentally change the outcomes.

an explosive increase in their ratio. This highlights the importance of employing right-tailed unit root tests on price-to-rent ratio, as they offer more insight into rational bubbles compared to tests solely focused on real prices. [2].

2.3 Explosive Bubble Tests

To verify the presence of explosive bubbles in the series, we will employ a methodology similar to that proposed by B. Diba and H. Grossman (1984) and Hamilton and Whiteman (1985). Firstly, to identify the order of integration of the series, we will use the Augmented Dickey-Fuller (ADF) test [3], whose equation is given by (Dickey and Fuller 1979, 1981):

$$\Delta y_t = \alpha_{r_1,r_2} + \gamma_{r_1,r_2} y_{t-1} + \sum_{k=1}^{p} \psi^k_{r_1,r_2} \Delta y_{t-k} + \epsilon_t \tag{8}$$

where y_t is the series of interest, r_1 and r_2 denote fractions of the total sample size that specify the starting and ending points of a subsample period, k is the maximum number of lags included in the specification, α_{r_1,r_2}, γ_{r_1,r_2}, $\psi^k_{r_1,r_2}$ are parameters, and ϵ_t is white noise. We will select the number of lags p using the Akaike information criterion. With this test, we will evaluate the hypotheses:

$$H_0 : \gamma_{r_1,r_2} = 0$$
$$H_a : \gamma_{r_1,r_2} < 0$$

If the null hypothesis is rejected, it implies that the parameter $(1+\gamma_{r_1,r_2})$ is within the unit circle, indicating that the series is stationary and the absence of explosive bubbles. To conduct this hypothesis test, we can construct the test statistic as follows:

$$\mathrm{ADF}^{r_2}_{r_1} = \frac{\hat{\gamma}_{r_1,r_2}}{s.e.(\hat{\gamma}_{r_1,r_2})} \tag{9}$$

where $s.e.(\hat{\gamma}_{r_1,r_2})$ denotes the estimate of the standard deviation of the estimated parameter $\hat{\gamma}_{r_1,r_2}$. Establishing the values of r_1 as 0 and r_2 as 1 results in the conventional ADF test, ADF^1_0. Such as present in Paparoditis and Politis (2016), the asymptotic distribution of ADF^1_0 under H_0 is given by:

$$\frac{\int_0^1 W\,dW}{(\int_0^1 W^2)^{1/2}} \tag{10}$$

where W is the standard Wiener process. In a second step, if we cannot reject H_0, we will utilize the cointegration test proposed by Engle and Granger (1987) and the one introduced by Johansen (1988, 1995) to assess the presence of cointegration between the price series and the rents. If there is evidence in favor of cointegration, it indicates the absence of explosive behavior, allowing us to rule out the presence of explosive bubbles.

[2] Even if there is apparent explosive behavior in these observable ratios, it's essential to recognize that this might not conclusively rule out the possibility that the explosiveness stems from the unobserved component of fundamentals. This inherent challenge is pervasive in nearly all empirical studies, highlighting the complexity of accurately assessing the presence of bubbles.

[3] Davidson and MacKinnon (2004) findings indicate that, in finite samples, the Augmented Dickey-Fuller (ADF) test outperforms the P. C. B. Phillips and Perron (1988) test for unit root diagnostics.

However, the aforementioned tests do not detect the explosive behavior of the series. For this purpose, we will use a right-tailed *Augmented Dickey-Fuller* test, which tests the same parameter γ in expression (8) for the difference between price and fundamental and price-to-rent ratio, altering the hypotheses being tested. In this test, we will evaluate the hypotheses:

$$H_0 : \gamma_{r_1,r_2} = 0$$
$$H_a : \gamma_{r_1,r_2} > 0$$

If the null hypothesis is rejected, it indicates that the term $1 + \gamma_{r_1,r_2}$ is outside the unit circle, suggesting an explosive process in the series. We can evaluate this hypothesis using the same construction expressed in (9), conducting a right-tailed ADF test. We will apply the test to price-to-rent ratio series. This approach will allow us to assess whether the series exhibits explosive behavior, indicating a deviation between the observed price and the estimated fundamental value. It will highlight one episode of explosiveness in the entire series tested.

Nevertheless, because the standard ADF test lacks consistency with changes in regime, it demonstrates extremely low power in the presence of boom-bust episodes. Indeed, nonlinear dynamics, like those demonstrated by periodically collapsing speculative bubbles, frequently lead to the identification of false stationarity, even when the underlying process is inherently explosive (Evans 1991). Therefore, we will proceed with the investigation using other methods for bubble identification.

2.4 Periodically Collapsing Bubbles

A second type of bubbles that can be tested is the phenomenon of periodically collapsing bubbles. Throughout history, it has been observed that bubbles tend to be temporary, featuring alternating phases of asset price expansion and contraction. Various models have been introduced to understand these fluctuations. Notable examples encompass the early probabilistic bubble generation frameworks presented by M. W. Blanchard W. (1982) and Evans (1991). In the context of Blanchard and Watson's (1982) framework, bubbles are generated based on the following model:

$$B_{t+1} = \begin{cases} (\pi\beta)^{-1} B_t + \epsilon_{t+1}; & \text{with probability } \pi \\ \epsilon_{t+1}; & \text{with probability } 1 - \pi \end{cases}$$

Where ϵ_{t+1} represents an error term. Within this model, the probability of a bubble occurring in each time period is predetermined as π, while the probability of a bubble collapse is designated as $1 - \pi$. The expansion rate of the bubble assumes the autoregressive parameter $(\pi\beta)^{-1}$ throughout the expansion phase of the bubble. The conditional expectation of B_{t+1} satisfies the submartingale property the and can be expressed as follows:

$$E_t[B_{t+1}] = E_t[\pi((\pi\beta)^{-1} B_t + \epsilon_{t+1}) + (1 - \pi)\epsilon_{t+1}] = \beta^{-1} B_t \tag{11}$$

Differently from M. W. Blanchard W. (1982), Evans (1991) examines a three-stage bubble model. In the initial phase, the bubble experiences growth at a rate of β^{-1}. Once surpassing the threshold α, the rate of expansion intensifies to $(\pi\beta)^{-1}$, and within this interval, there exists a chance of a subsequent collapse. The process of collapse is regulated by a Bernoulli process denoted as θ_t, where the value one is assumed with a probability of π, and zero otherwise. To elaborate further,

$$B_{t+1} = \begin{cases} \beta^{-1} B_t u_{t+1}; & \text{if } B_t \leq \alpha \\ \left(\delta + \pi^{-1}\beta^{-1}\theta_{t+1}\left(B_t - \beta\delta\right)\right) u_{t+1}; & \text{if } B_t > \alpha \end{cases} \tag{12}$$

where δ and α are parameters satisfying $0 < \delta < \beta^{-1}\alpha$ and takes the forms of $u_{t+1} = exp(v_{t+1} + w^2/2)$ with $v_{t+1} \overset{i.i.d.}{\sim} N(0, w^2)$. This guarantees that:

$$E_t[B_{t+1}] = \beta^{-1} B_t \tag{13}$$

The intuition behind the process in the equation (12) is as follows: while $B_t \leq \alpha$, the bubble grows at a mean rate β^{-1} and the probability of collapse is null. However, if $B_t > \alpha$, the bubble begins to grow at a higher rate and the new mean growth rate will be $\beta^{-1}\pi^{-1}$. In this case, the bubble may collapse with probability $1 - \pi$. Once this occurs B_t does not vanish, rather it takes a (mean) value δ and then the process starts again (Evans 1991).

A formal method for measuring an asymmetric adjustment process, extending the Dickey–Fuller test, is presented in the MTAR[4] model proposed by Engle and Granger (1987) and Enders and Siklos (2001). First, it is estimated the following regression and the residuals calculated:

$$P_t = \hat{\imath} D_t + \hat{\mu}_t \tag{14}$$

The threshold for the change in the dynamics is adjusted by the following regression:

$$\Delta\hat{\mu}_t = I_t \rho_1 \hat{\mu}_{t-1} + (1 - I_t)\, \rho_2 \hat{\mu}_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta\hat{\mu}_{t-i} + \epsilon_t \tag{15}$$

in that equation, the indicator function I_t is defined by:

$$I_t = \begin{cases} 1; & \text{if } \Delta\hat{\mu}_{t-1} \geq \tau \\ 0; & \text{if } \Delta\hat{\mu}_{t-1} < \tau \end{cases} \tag{16}$$

The conventional formulation of the MTAR test, assuming the value of the threshold as $\tau = 0$. However Chan (1993) proposed a methodology to estimate it. The procedure involves discarding the top 15% and bottom 15% of the residuals, estimating the MTAR models for the remaining values, and selecting for the second model

[4] Just as indicate by Bohl (2003), Enders and Siklos (2001) also examine the characteristics of threshold autoregressive (TAR) adjustment, where equation (16) depends on the level of the estimated residuals $\hat{\mu}_{t-1}$ instead of first diference $\Delta\hat{\mu}_{t-1}$. However, it's worth noting that the TAR test generally exhibits lower statistical power compared to the conventional Engle-Granger test. Additionally, the TAR adjustment model is unable to capture the distinct, asymmetrically rapid adjustments toward the long-run equilibrium that are often observed in the context of periodically collapsing bubbles.

the threshold that results in the lowest residual sum of squares [5].

Two consecutive hypothesis tests are conducted to investigate the potential presence of a bubble. Firstly, using the F-statistic, we examine the null hypothesis of no cointegration, denoted as:

$$H_0 : \rho_1 = \rho_2 = 0$$
$$H_a : \rho_1 < 0 \text{ or } \rho_2 < 0$$

Not rejecting the null hypothesis may be indicative of a cointegrating relationship between P_t and D_t [6]. Else if the null hypothesis is rejected in the first test, we proceed as suggested by Enders and Siklos (2001) with a second test to investigate the presence of symmetry, represented by the hypothesis:

$$H_0 : \rho_1 = \rho_2$$
$$H_a : \rho_1 \neq \rho_2$$

This second test employs the non conventional F-statistic. If the null hypothesis is rejected, we conduct an additional test. If the estimated coefficient $\hat{\rho_1}$ is statistically significant, negative, and greater in absolute terms than the estimated $\hat{\rho_2}$, we reject the hypothesis of symmetric adjustment. In such a scenario, there is evidence suggesting the presence of periodically collapsing bubbles. The critical values for the t and F statistics for the cointegration test and the assymetric test are provided by Enders and Siklos (2001).

The MTAR technique is specifically designed for empirically identifying periodically collapsing bubbles. The theoretical framework posits the potential for positive bubbles, but not negative ones, emphasizing a distinct pattern of stock price increases in relation to dividends before a market crash (Bohl 2003). This suggests an inherent asymmetry in the behavior of the residual of the cointegrating regression in the equation (12). Periodically collapsing bubbles are discerned through shifts in $\Delta\hat{\mu}_{t-1}$ surpassing the threshold, followed by a sharp decline to the threshold level. Conversely, instances of changes in $\Delta\hat{\mu}_{t-1}$ below the threshold do not exhibit bubble eruptions followed by a collapse.

To illustrate, consider a threshold of $\tau = \bar{\tau}$ in equation (16) . A value of $\Delta\hat{\mu}_{t-1} > \bar{\tau}$ indicates a surge in asset prices relative to returns followed by a crash, while a comparable behavior to $\Delta\hat{\mu}_{t-1} < \bar{\tau}$ resulting in a sharp increase back to the equilibrium position is not expected. This establishes an asymmetry in deviations from the equilibrium, signifying the presence of periodically collapsing bubbles. Consequently, if the estimated coefficient $\hat{\rho_1}$ is both statistically significant and negatively larger in absolute terms compared to the parameter $\hat{\rho_2}$, the null hypothesis of symmetric adjustment ($H_0 : \rho_1 = \rho_2$) is rejected. The rejection of this null hypothesis provides evidence supporting the existence of periodically collapsing bubbles in asset prices.

2.5 Multiple Bubble Test

To assess the presence and date multiple bubbles episodes, we will utilize more general univariate right-tailed test, outline the associated date-stamping strategies, and discuss technical details about their implementation.

[5] More details on the choice of those values can be seen in Enders and Siklos (2001).

[6] The Engle and Granger (1987) can be seen as a specific instance within the broader framework of the MTAR model. As described by Enders and Siklos (2001), within a reasonable range of adjustment parameters, the MTAR test can possess significantly greater statistical power compared to the Engle-Granger test, especially in cases where there are asymmetric departures from equilibrium there is present (Bohl 2003).

The *Generalized Supremum Augmented Dickey Fuller (GSADF)* test, introduced by P. C. B. Phillips, Shi, and Yu (2015a,b), extend the right-tail ADF test covering a larger number of subsamples and testing the null hypothesis of a unit root and the alternative is of a mildly explosive process[7].

Detailing the estimation process, let's consider a minimum window size $r_0 \in (0, 1)$. The methodology involves estimating the regression (8) for all possible subsamples of size r_0 by allowing both the starting point, $r_1 \in [0, r_0]$, and the ending point, $r_2 \in [r_0, 1]$, to change. The increased flexibility in the estimation window leads to significant improvements in power and enhances the suitability of the GSADF test for identifying multiple changes in regime. Thus, based on equation (8), we can define the GSADF statistic as:

$$GSADF(r_0) \equiv \sup_{\substack{r_2 \in [r_0,1] \\ r_1 \in [0, r_2 - r_0]}} \left\{ \mathrm{ADF}_{r_1}^{r_2} \right\} \tag{17}$$

and its limit distribution under the null is:

$$\mathrm{GSADF}(r_0) \xrightarrow{d} \sup_{\substack{r_2 \in [r_0,1] \\ r_1 \in [0, r_2 - r_0]}} \left\{ \frac{\frac{1}{2}(r_2 - r_1)\left[W(r_2)^2 - W(r_1)^2 - (r_2 - r_1)\right] - \int_{r_1}^{r_2} W(s)ds\left[W(r_2) - W(r_1)\right]}{(r_2 - r_1)^{\frac{1}{2}}\left[(r_2 - r_1)\int_{r_1}^{r_2} W(s)^2 ds - \left(\int_{r_1}^{r_2} W(s)ds\right)^2\right]^{\frac{1}{2}}} \right\}$$

where $W(.)$ denotes the standard Wiener process. These results are presented in the article by Monschang and Wilfling (2021). Once more, rejecting the unit root hypothesis in favor of explosive behavior necessitates the test statistic surpassing the right-tail critical value derived from its limit distribution.

Furthermore, if the null hypothesis of a unit root in is rejected, the utilization of a date-stamping strategy enables us to estimate the origination and termination dates of exuberance period according, respectively, by the following expression:

$$\hat{r}_e = \inf_{r_2 \in [r_0,1]} \left\{ r_2 : BSADF_{r_2}(r_0) > scu_{r_2}^{\alpha} \right\} \tag{18}$$

$$\hat{r}_f = \inf_{r_2 \in [r_e,1]} \left\{ r_2 : BSADF_{r_2}(r_0) < scu_{r_2}^{\alpha} \right\} \tag{19}$$

Where the value $scu_{r_2}^{\alpha}$ represents the critical threshold for the statistic, considering $\lfloor r_2 T \rfloor$ observations at a significance level of $100(1-\alpha)\%$. And the Backward Sup Augmented Dickey Fuller (BSADF) statistics[8] used to date the exuberance periods can be define as:

$$BSADF_{r_2}(r_0) \equiv \sup_{r_1 \in [0, r_2 - r_0]} \left\{ \mathrm{ADF}_{r_1}^{r_2} \right\} \tag{20}$$

[7] P. C. Phillips and Magdalinos (2007) introduce the concept of a mildly explosive root through the following data generating process:

$$y_t = \delta_n y_{t-1} + \epsilon_t,$$

where $\delta_n = 1 + \frac{c}{k_n}$, and $(k_n)_{n \in \mathbb{N}}$ is a sequence that grows to ∞ in a way such that $k_n = o(n)$ as $n \to \infty$. The development of limit theory for mildly explosive processes is presented in P. C. Phillips and Magdalinos (2007).

[8] The BSADF statistic is connected to the GSADF statistic through the following relationship:

$$GSADF(r_0) \equiv \sup_{r_2 \in [r_0,1]} \left\{ \mathrm{BSADF}_{r_2}(r_0) \right\}$$

Delving deeper into the implementation technique, to determine the minimum window size, we employ the expression $r_0 = T(0.01 + 1.8/\sqrt{T})$ as recommended by the rule proposed by P. C. B. Phillips, Shi, and Yu (2015a) and P. C. B. Phillips, Shi, and Yu (2015b), with the lag parameter set to $k = 0$.[9]. Regarding the choice of parameter k, simulations provide evidence that the suggested methodologies for detecting right-tailed unit roots perform effectively when the number of lags is set to a low value, specifically 0 or 1. In contrast, selecting lags based on information criteria can lead to significant distortions in the test's size (Vasilopoulos, E.G. Pavlidis, and Martínez-García 2020).

Additionally, concerning the null hypothesis, the GSADF statistic converges and adheres to the distribution outlined by P. C. B. Phillips, Shi, and Yu (2014). To ascertain the distribution of our test statistic, we utilize the bootstrap procedure introduced by (P. Phillips and Shi 2020). This method involves a wild bootstrap re-sampling scheme, which is asymptotically robust to non-stationary volatility.

2.6 Panel Multiple Bubble Tests

Additionally, to jointly evaluate the presence of multiple bubbles at municipal levels, we employ the panel GSADF test proposed by Efthymios Pavlidis, Yusupova, et al. (2016). This test extends the conventional GSADF procedure to accommodate heterogeneous panels and allow us to date episodes of overall exuberance (Vasilopoulos, E.G. Pavlidis, and Martínez-García 2020). The panel version of this test can be expressed by the equation:

$$\Delta y_{i,t} = \alpha_{i,r_1,r_2} + \gamma_{i,r_1,r_2} y_{i,t-1} + \sum_{k=1}^{p} \psi^k_{i,r_1,r_2} \Delta y_{i,t-k} + \epsilon_{i,t} \tag{21}$$

where $i = 1, ..., N$, denotes the municipality index, and the remaining variables are defined as in the previous sub-section. We are interested in testing the null hypothesis of a unit root for all N cities against the alternative of explosive behavior in a subset of cities units, that can be express by:

$$H_0 : \gamma_{i,r_1,r_2} = 0$$
$$H_a : \gamma_{i,r_1,r_2} > 0$$

This alternative allows for γ_{i,r_1,r_2} to differ across panels and, in that sense, is more general than approaches that impose a homogeneous alternative hypothesis.The testing procedure involves averaging the individual BSADF statistics at each time period:

$$BSADF_{i,r_2}(r_0) \equiv \sup_{r_1 \in [0, r_2 - r_0]} \left\{ \text{ADF}^{r_2}_{r_1} \right\} \tag{22}$$

And the panel BSADF can be define as:

$$\text{Panel } BSADF_{r_2}(r_0) = \frac{1}{N} \sum_{i=1}^{N} BSADF_{i,r_2}(r_0) \tag{23}$$

[9]Opting for a fixed lag length is advantageous as it enables the utilization of a recursive least squares methodology, significantly lowering the computational cost of estimation (Paparoditis and Politis 2016).

In a similar way to how we define the GSADF in equation (17), we can define the panel GSADF by applying the supremum operator, yielding the following:

$$\text{Panel } GSADF(r_0) \equiv \sup_{\substack{r_2\in[r_0,1]\\ r_1\in[0,r_2-r_0]}} \text{Panel } BSADF_{r_2}(r_0) \tag{24}$$

The results of Chang (2004) and Maddala and Wu (1999) show that the distribution of panel unit root tests based on mean unit root statistics is not invariant to cross-sectional dependence of the error terms, ϵ_i. In light of the ample evidence of strong financial linkages across countries (Milesi-Ferretti and Lane 2003), the assumption of uncorrelated shocks seems unrealistic even for national commercial real estate markets. In order to draw inferences in this context, we adopt a sieve bootstrap approach that is designed specifically to allow for cross-sectional error dependence, as propose by Efthymios Pavlidis, Yusupova, et al. (2016).

2.7 Intrinsic Bubble Tests

Intrinsic bubbles, as introduced by Froot and Obstfeld (1991), propose that the moments of exuberance in asset prices may be driven by fundamentals in a non-linear manner. This implies a non-linear relationship between changes in asset prices and fundamentals, the intrinsic factor, thereby distinguishing intrinsic bubbles from their rational counterparts.

Therefore, to implement the intrinsic bubble test we consider the intrinsic bubbles are a function of rental price adheres to the following expression:

$$B_t \equiv bD_t^{\lambda} \tag{25}$$

where b is an arbitrary constant and λ is the positive root of a quadratic equation $\lambda\mu + \frac{\lambda^2\sigma^2}{2} + \log\beta = 0$ [10]. In the section 2.2, we showed that fundamental value of commercial real estate prices is proportional to the rental price during period t:

$$P_t^f = \kappa D_t \tag{26}$$

Where $\kappa = \left(\exp\left(-\log\beta\right) - \exp\left(\mu + \frac{\sigma^2}{2}\right)\right)^{-1}$. So, in the presence of a bubble, the present value of the prices can be obtain by add the bubble component:

$$P_t = \kappa D_t + bD_t^{\lambda} \tag{27}$$

Under this setup, the inequality $\mu + \frac{\sigma^2}{2} + \log(\beta) < 0$, assumed in the section 2.2, implies that λ must be greater than 1, and it is an explosive nonlinear relation between bubbles and the rental price index, so the property price index may overreact to information about the rental price index. And the intrinsic bubble definition satisfy the martingale condition:

[10] This bubble definition 25 is derive of the rational bubble equation present in (7) and the hypothesis of $d_t = \log(D_t)$ is a random walk process.

$$\beta E_t[B_{t+1}] = \beta E_t[bD_{t+1}^{\lambda}] = \beta E_t[bD_t^{\lambda} e^{\lambda(\mu+\varepsilon_{t+1})}] =$$
$$= \beta(bD_t^{\lambda} e^{\lambda\mu + \frac{\lambda^2\sigma^2}{2}}) = \beta(\beta^{-1} bD_t^{\lambda}) = B_t \tag{28}$$

Then, for our empirical approach to avoid collinearity among the explanatory variables, we divide the equation by D_t as suggest by (Froot and Obstfeld 1991):

$$\frac{P_t}{D_t} = c_0 + c_1 (D_t)^{\lambda-1} + \xi_t \tag{29}$$

When conducting bubble tests, the null hypothesis of no bubble posits that $c_1 = 0$, while the alternative hypothesis suggests that $c_0 = k$ and $c_1 > 0$, indicating the presence of a bubble.

3 Data

For our empirical research, we use data on average prices per square meter and average rental prices per square meter for Brazilian commercial real estate market. To obtain the real prices, we deflate the prices using the Whole National Consumer Price Index (IPCA). The IPCA is a widely used inflation index in Brazil, reflecting the average price changes experienced by Brazilian consumers[11].

Data source on sale price and rental price indices come from the FIPEZAP survey made available by the Institute of Economic Research Foundation (FIPE), and data of deflate index is provide by Brazilian Institute of Geography and Statistics (IBGE). To conduct the analysis, we used the full-sample for all cities for period available from 2012 to 2023 in monthly frequency for real prices[12].

4 Empirical Results

In this section, we present the empirical results of the bubble test for the Brazilian commercial real estate market.

4.1 Explosive Bubble Tests

Firstly, to check the presence of explosive bubbles, we will identify the order of integration of average real prices per square meter and average rental real prices per square meter for Brazilian commercial real estate market using the Augmented Dickey-Fuller (ADF) test with constant, whose equation is provided in equation (8). To determine the appropriate lags for the test, we will utilize the Aikaike information criterion (AIC). The obtained results are presented in Table 1 [13].

[11] The month of January 2012 has been deemed the reference month in this research.

[12] The national index and the cities of São Paulo and Rio de Janeiro present data for the period from Jan. 2012 to Oct. 2023. For Belo Horizonte, data is available from Dec. 2013 to Oct. 2023. In Porto Alegre, data covers the period from Dec. 2015 to Oct. 2023. Lastly, the database provides information for the cities of Campinas, Brasília, Salvador, Curitiba, and Florianópolis for the period from Jan. 2018 to Oct. 2023.

[13] The result in Table 1 pertains to P_t and D_t in levels. A similar procedure was conducted for the logarithmic series, yielding similar outcomes for both prices and rents (Appendix 6.1 provides a more detailed account of these results).

Table 1: Results from the ADF unit root test.

	Variable	ADF Level	ADF 1° Diff	Integration order
Panel A: data for the Brazil				
Brazil	Price	0.146	-3.911***	$I(1)$
	Rent	-1.446	-4.763***	$I(1)$
Panel B: data for the cities				
São Paulo	Price	-0.310	-5.017***	$I(1)$
	Rent	-1.751	-5.740***	$I(1)$
Rio de Janeiro	Price	0.112	-3.488***	$I(1)$
	Rent	0.600	-5.037***	$I(1)$
Belo Horizonte	Price	-1.234	-7.444***	$I(1)$
	Rent	-4.414	-5.206***	$I(1)$
Porto Alegre	Price	0.357	-5.918***	$I(1)$
	Rent	-0.896	-4.894***	$I(1)$
Brasília	Price	-0.810	-4.267***	$I(1)$
	Rent	-0.464	-6.392***	$I(1)$
Campinas	Price	-0.863	-5.052***	$I(1)$
	Rent	-2.135	-3.473**	$I(1)$
Curitiba	Price	-2.494	-4.218***	$I(1)$
	Rent	-1.265	-4.408***	$I(1)$
Florianópolis	Price	-1.826	-4.787***	$I(1)$
	Rent	-1.801	-3.061**	$I(1)$
Salvador	Price	-0.615	-4.951***	$I(1)$
	Rent	-1.440	-5.460***	$I(1)$

Notes: The table presents the the Augmented Dickey-Fuller with constant and no linear trend for the real price and real rent series.
*** Indicates statistical significance at the 1% level.
** Indicates statistical significance at the 5% level.
* Indicates statistical significance at the 10% level.

Both tests presented above enable us to conclude that both the real price and the real rent follow an integrated process of first order $I(1)$ for all series. In a subsequent step, we will employ the test of Engle and Granger (1987) as well as the test of Johansen (1988, 1995) to assess the cointegration between the price series and the underlying fundamentals - rents. If these series display cointegration, this will give us some evidence in the sense of the absence of explosive behavior, thereby allowing us to dismiss the possibility of explosive bubbles. The table 2 provides the outcomes of the Engle and Granger (1987) test and the Johansen (1988, 1995) cointegration test for the prices and rents series.

The results presented in the cointegration tests do not provide sufficient evidence to assert the cointegration of the series with their fundamentals for the series of Porto Alegre, Brasília, Campinas, Florianópolis and Salvador. Consequently, the possibility of a explosive bubble in series cannot be ruled out (B. T. Diba and H. I. Grossman 1988). In the case of Belo Horizonte, both tests suggest cointegration and exclude the presence of a explosive bubble. For the Brazil price index series and the cities of São Paulo, Rio de Janeiro, and Brasília, the results from the two tests are conflicting: the Engle–Granger test indicates no cointegration and suggests the possibility of a explosive bubble, while Johansen's test, with a certain level of confidence, suggests they cointegrate and rules out the presence of explosive bubble.

Since the preceding tests have not dismissed the bubble hypothesis for certain series, we will employ a right-tailed Augmented Dickey-Fuller (ADF) test. This test examines the same parameter γ_{r_1,r_2} from equation (8)

Table 2: Results from Cointegration test

	Cointegration	Engel-Granger	Johanssen		
			Null	λ_{trace}	λ_{Max}
Panel A: data for the Brazil					
Brazil	Yes	0.369	r = 0	32.11^{***}	24.42^{***}
			$r \leq 1$	7.68^{*}	7.68^{*}
Panel B: data for the cities					
São Paulo	No	-2.54	r = 0	15.96^{*}	12.77
			$r \leq 1$	3.19	3.19
Rio de Janeiro	Yes	-0.803	r = 0	18.74^{**}	13.18^{*}
			$r \leq 1$	5.56	5.56
Belo Horizonte	Yes	-3.051^{***}	r = 0	19.70^{***}	24.27^{***}
			$r \leq 1$	4.58	4.58
Porto Alegre	No	-1.90	r = 0	12.68	12.63
			$r \leq 1$	0.05	0.05
Brasília	No	-2.31	r = 0	13.58	13.33^{*}
			$r \leq 1$	0.25	0.25
Campinas	No	0.621	r = 0	8.50	7.03
			$r \leq 1$	1.47	1.47
Curitiba	No	-1.49	r = 0	8.44	6.28
			$r \leq 1$	2.16	2.16
Florianópolis	No	0.128	r = 0	8.04	6.57
			$r \leq 1$	1.47	1.47
Salvador	No	0.016	r = 0	4.41	3.60
			$r \leq 1$	0.81	0.81

Notes: The table presents the the Engel-Granger test with no trend and Johanssen cointegration test with no intercept and no trend for the real price and real rent time series. The second column indicates whether the hypothesis of no cointegration hypothesis is rejected for one or both series.
*** Indicates statistical significance at the 1% level.
** Indicates statistical significance at the 5% level.
* Indicates statistical significance at the 10% level.

for the difference between price and fundamental series, while altering the tested hypotheses.This approach will enable us to evaluate whether the series displays explosive behavior, indicating a deviation between the observed price and the estimated fundamental value throughout the entire series. The outcomes of the right-tailed ADF test are summarized in Table 3.

Table 3: Results from the right-tailed Augmented Dickey-Fuller test to price-to-rent ratio.

	Bubble	Right-tail ADF
Panel A: data for the Brazil		
Brazil	No	-1.979
Panel B: data for the cities		
São Paulo	No	-1.586
Rio de Janeiro	No	-1.65
Belo Horizonte	No	-2.222
Porto Alegre	Yes	0.911^{***}
Brasília	Yes	-2.49
Campinas	Yes	0.048^{**}
Curitiba	No	-2.04
Florianópolis	Yes	-0.273^{*}
Salvador	No	-1.283

Notes: The table shows the right-tail Augmented Dickey-Fuller test statistics and indicate their corresponding significance levels for the time series of real prices and fundamental difference and price-to-rent ratio. The second column indicates whether the hypothesis of the existence of a bubble is accepted for one or both series.
*** Indicates statistical significance at the 1% level.
** Indicates statistical significance at the 5% level.
* Indicates statistical significance at the 10% level.

As shown in Table 3, the null hypothesis is rejected in the price-to-rent ratio for the cities of Porto Alegre, Campinas and Florianópolis. This finding aligns with the results presented in the integration and cointegration tests discussed earlier, which did not provide evidence of cointegration between the rent and price. Nevertheless, for the remaining cities, we do not observe the existence of an explosive bubble when applying the right tail ADF test to the entire available data sample.

4.2 Periodically Bubble Tests

To delve beyond the tests proposed by B. Diba and H. Grossman (1984) and Hamilton and Whiteman (1985), the research aims to investigate the presence of bubbles that collapse periodically, as formulated by Evans (1991). To ascertain the presence of periodically collapsing bubbles in Brazilian commercial real estate market, we employ the consistent estimate Momentum Threshold Autoregressive model (MTAR) as elaborated in Section 2.4. Table 4 displays the estimated parameters for the model.

Table 4: Results from the consistent estimate MTAR model.

Variable	Bubble	ι	τ	ρ_1	ρ_2	*Lag*	*AIC*	F_C	F_A
Panel A: data for the Brazil									
Brazil	No	219.16***	-1.13	-0.01	-0.09***	1	577.36	5.62***	7.38***
		(16.478)		(0.011)	(0.027)				
Panel B: data for the cities									
São Paulo	No	216.50***	1.19	0.02	-0.04***	1	576.84	3.94**	3.71*
		(13.165)		(0.025)	(0.014)				
Rio de	No	226.66***	1.42	0.01	-0.02**	4	623.83	2.81*	2.53
Janeiro		(24.773)		(0.016)	(0.010)				
Belo	No	230.28***	-1.08	-0.02	-0.12***	1	530.77	8.00***	8.79***
Horizonte		(16.059)		(0.016)	(0.032)				
Porto	No	240.07***	-1.41	0.01	-0.11***	0	528.58	3.96**	5.76**
Alegre		(19.404)		(0.029)	(0.041)				
Brasília	No	213.04***	-3.09	-0.12***	0.07	4	349.70	3.92**	2.66
		(18.213)		(0.045)	(0.109)				
Campinas	No	192.04***	-1.55	0.02	-0.29***	1	336.34	12.40***	19.91***
		(10.758)		(0.039)	(0.059)				
Curitiba	No	280.74***	-2.42	-0.02	-0.19***	4	380.06	6.73***	6.79**
		(20.271)		(0.039)	(0.053)				
Florianópolis	No	232.17***	-1.37	-0.01***	-0.15	2	358.33	4.97**	4.81**
		(13.361)		(0.043)	(0.049)				
Salvador	No	150.93***	-1.60	-0.03	-0.25***	1	346.56	7.72***	6.47**
		(8.721)		(0.058)	(0.064)				

Notes: The table presents the parameters estimate by the consistent momentum threshold autoregressive model (MTAR). F_C and F_A represent the F-statistics used to test the null hypotheses concerning a unit root, specifically $H_0 : \rho_1 = \rho_2 = 0$, and symmetry, denoted as $H_0 : \rho_1 = \rho_2$, respectively. The lag for each test is chosen through the general-to-specific approach (Ng and Perron 1995), permitting a maximum lag order of 4.
*** Indicates statistical significance at the 1% level.
** Indicates statistical significance at the 5% level.
* Indicates statistical significance at the 10% level.

In Table 4, the MTAR test reveals no evidence of periodically occurring bubbles, as formulated by Evans (1991), across all series considered throughout the entire sample period. The MTAR test also indicates the presence of cointegration between the price and rent series at a significance level equal to or greater than 10% for all series, rejecting the null hypothesis $H_0 : \rho_1 = \rho_2 = 0$. Additionally, the null hypothesis of symmetrical adjustment ($H_0 : \rho_1 = \rho_2$) is rejected for all series at a significance level equal to or greater than 10%, except for Rio de Janeiro and Brasília, ruling out the possibility of asymmetric adjustment to variations in $\hat{\mu}_t$ for these two series[14].

[14]Enders and Siklos (2001) demonstrate that within a reasonable range of adjustment parameters, the power of the MTAR test

Upon further investigation into the presence of periodically collapsing bubbles, it is observed that the condition, wherein the estimated $\hat{\rho_1}$ is significantly negative and its absolute value exceeds the estimate of $\hat{\rho_2}$, is not met for the remaining series. Consequently, we did not find evidence suggesting the presence of periodically collapsing bubbles. Nonetheless, the MTAR test enhanced our comprehension of the dynamics of $\hat{\mu_t}$, unveiling an asymmetric regime of adjustment when crossing the threshold.[15].

4.3 Multiple Bubble Tests

To enhance the accuracy of identifying phases of exuberance in the Brazilian commercial real estate market, we employ the Generalized Sup Augmented Dickey-Fuller (GSADF) test. In this context, we apply the test to the real prices and the price-to-rent ratio, allowing us to evaluate the presence of exuberance periods in the both series. This approach enables us to uncover unsustainable trends in the asset's rate of return, highlighting instances of divergence between observable real prices and fundamental prices.

Table 5: Results for the univariate GSADF test and panel GSADF panel test.

	Bubble	Real prices	Price-to-rent ratio
Panel A: data for the Brazil			
Brazil	Yes	6.53^{***}	5.44^{***}
Panel B: data for the cities			
São Paulo	Yes	5.97^{***}	3.78^{***}
Campinas	Yes	2.90^{***}	5.54^{***}
Rio de Janeiro	Yes	3.71^{***}	3.31^{***}
Belo Horizonte	Yes	4.66^{**}	2.86^{**}
Brasília	No	2.71^{***}	-0.02
Salvador	Yes	3.13^{**}	3.35^{**}
Porto Alegre	Yes	2.70^{***}	2.46^{***}
Curitiba	No	1.57	1.22
Florianópolis	Yes	2.41^{**}	2.30^{***}
Panel C: panel data for the cities			
Panel GSADF statistic[16]	Yes	1.97^{***}	1.93^{***}

Notes: The table shows the GSADF test statistics and indicate their corresponding significance levels for the time series of real prices and real rents. The findings pertain to an autoregressive lag length of $k = 0$ in all cases. The second column indicates whether the hypothesis of the existence of a bubble is accepted or not in the price-to-rent ratio.
*** Indicates statistical significance at the 1% level.
** Indicates statistical significance at the 5% level.
* Indicates statistical significance at the 10% level.

Table 5 reports the results of the GSADF tests on real prices and the price-to-rent ratio. A comparison of the results from the two methods reveals a high degree of similarity, providing strong evidence of the presence of

can surpass that of the Engle–Granger test when there are asymmetric departures from equilibrium. This result aligns with our empirical findings, as for every series, except Brasília and Rio de Janeiro, that rejected the null hypothesis of non-cointegration, they also rejected the null hypothesis of the absence of asymmetry in the second test. In other words, the MTAR was more effective in detecting cointegration than the conventional Engle–Granger test, as we observe the presence of asymmetric behavior in the residuals of equation (14).

[15] Articles such as Waters and J. E. Payne (2007) and Liu, Hammoudeh, and Thompson (2013) advocate for the possibility of the existence of a negative bubble and investigate the presence of periodically collapsing bubbles using a similar MTAR methodology. In this scenario, ρ_2 must be significantly negative, and its absolute value exceeds the estimate of ρ_1, indicating an asymmetrical adjustment back to equilibrium. The adjustment is faster following a negative shock, i.e., increasing faster to the equilibrium. In our empirical results, the parameter estimates for Brazil, São Paulo, Belo Horizonte, Porto Alegre, Campinas, Curitiba, and Salvador support this pattern. The outcome suggests that investors could potentially exploit distinct profit opportunities by leveraging the asymmetric nature.

[16] The results for the panel GSADF correspond to the period between January 2018 and October 2023 for all cities in the sample.

exuberance periods in the Brazilian commercial real estate market. The null hypothesis is rejected at a significance level of 1% for the Brazil index, São Paulo, Campinas, Rio de Janeiro, Porto Alegre, and Florianópolis. In the real prices series, we obtain the same result except for Florianópolis, rejecting the null hypothesis at a significance level of 5%. The evidence supporting mildly explosive behavior remains robust for Belo Horizonte and Salvador when examining the real price and fundamentals and the price-to-rent ratio at a significance level of 5%. Moreover, for Brasília and Curitiba, the test shows no evidence of periods of exuberance in the series of price-to-rent ratio. For Brasília, the GSADF test in real prices indicates periods of exuberance. However, this explosive behavior is due to movements in rents, with no periods of exuberance observed in the price-to-rent ratio series.

This finding improves upon the results presented in the explosive bubble test conducted earlier, which did not provide evidence of exuberant periods for the Brazil index, São Paulo, Rio de Janeiro, Belo Horizonte, Florianópolis, and Salvador. Due to the superior power properties of the GSADF, it was not only possible to find evidence of the presence of bubbles for the aforementioned cities, but also to infer that instances of exuberance were widespread across commercial real estate markets.

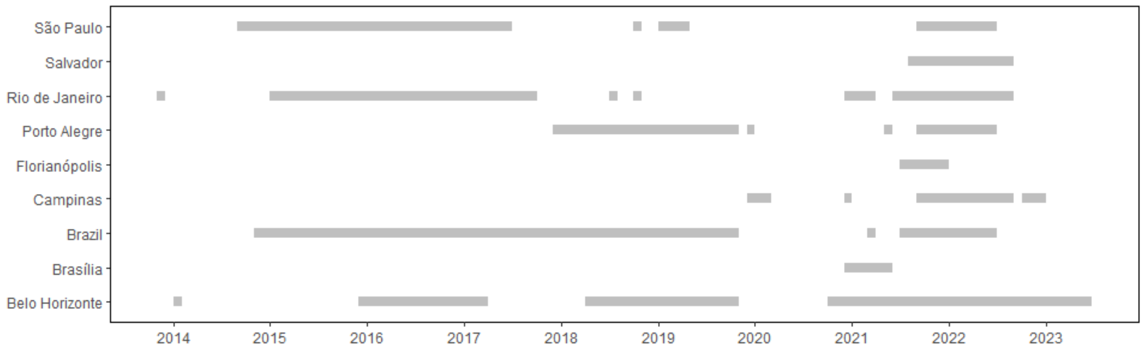

Figure 1: Date-stamping with real prices.

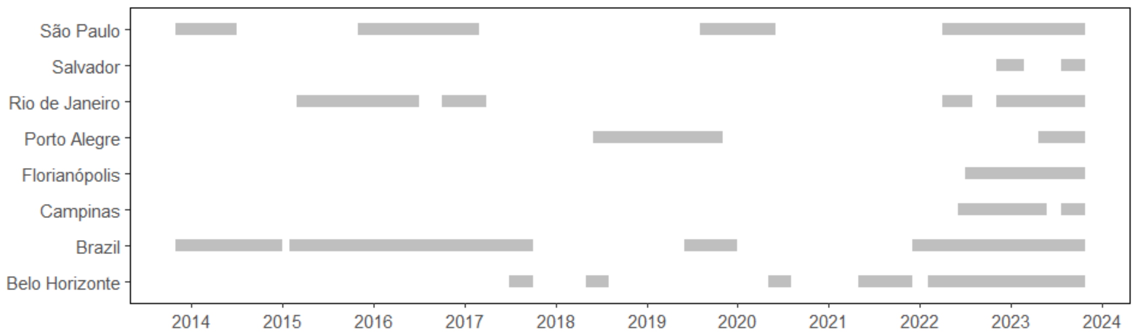

Figure 2: Date-stamping with price-to-rent ratios.

Dive into the timeline using the date-stamping strategy to identify exuberance periods. Figures 1 and 2 illustrate the periods during which the examined series displayed explosive dynamics, i.e., when the estimated BSADF statistics exceed the corresponding critical values at a confidence level of 90%. In the graphs, similarities can be observed for both the real prices and price-to-rent ratio, with shared periods of exuberance between the two methods noticeable for each city.

Looking at the results for real prices, we observe extended periods of exuberance indicating a rapid decline in real prices (Appendix 6.2). Notably, there has been a prolonged decline in prices in the cities of São Paulo, Belo Horizonte, and Rio de Janeiro during the period from 2015 to 2018. Moreover, it is crucial to highlight an explosively synchronized process in eight of nine cities beginning at the end of 2021. However, this result alone cannot be interpreted as conclusive evidence of the presence of a bubble, as presented by P. C. B. Phillips, Shi, and Yu (2015a) and Efthymios Pavlidis, Yusupova, et al. (2016). Some periods of price decline are accompanied by decreases in rents, while in others, there is a more substantial drop in rents than in prices, indicating a contrary movement in the bubble formation process. Therefore, a deeper analysis is required by examining the dynamics of the price-to-rent ratio series.

Examining the outcomes concerning price-to-rent ratios, we observe some commonalities with shared periods of exuberance between the two series, noticeable for each city. Perhaps not surprisingly, however, the periods of explosive dynamics in the price-to-fundamental ratios are somewhat shorter or pointing in the opposite direction due to the dynamics of rents. Given this consideration, it is possible to highlight two explosively synchronized processes in the series of price-to-rent ratio.

In the first phase, from early 2014 until the end of 2018, the price-to-rent ratio exhibits exuberance periods in the cities of São Paulo, Rio de Janeiro, Belo Horizonte and the Brazilian index. These periods are characterized by a decline in both prices and rents, but with a greater decrease in rents, leading to a reduction in the rate of return, as indicated by the growth in the price-to-rent ratio. This suggests a period of inflation of a bubble where prices do not align with the decline in fundamentals (Appendix 6.3).

A plausible reason for the formation of this bubble might be linked to a period of economic recession in Brazil with low growth and an increase in unemployment. Additionally, this period records a decline in investment in the construction sector as a reflection of reduced demand for real estate assets, both commercial and residential. In this way, considering the low elasticity of real estate supply, it could be a possible explanation for the reduction in the rate of return and the price-to-rent ratio.

A subsequent phase of exuberance can be dated between mid-2018 and the begining of 2020 for the cities of Porto Alegre, Campinas and São Paulo. This result is also reflected in the Brazil index, indicating the initial burst of a bubble in this cities. However, the second phase of synchronized exuberance, evident in our study, takes place from mid-2021 until the conclusion of the sample period. This latter phase is characterized by a synchronized trend in seven out of the nine cities in the dataset and the Brazil index, strongly suggesting the existence of a shared factor driving this dynamic. During this period, we witness both a decline in real prices and a reduction in the price-to-rent ratio (Appendix 6.2 and Appendix 6.3). This timeframe represents the deflation of a bubble, indicated by a more pronounced drop in prices compared to property rents.

The potential explanation for this second phase is associated with the resurgence of commercial activities and the relaxation of restrictions on circulation and trade, necessitated by the Covid-19 pandemic. This has led to the recovery of numerous in-person activities in major urban centers and a renewed patronage of commercial spaces. Moreover, in the same direction on the demand side, there is a decrease in the unemployment rate and a resumption of economic growth, thereby supporting a warming in the real estate market. Additionally, during the highlighted period, with an increase in the basic interest rate, there is a rationale for a decline in the price-to-rent ratio.

4.4 Panel Multiple Bubble Tests[17]

The findings from the panel of Brazilian cities suggest that the null hypothesis of a unit root can be rejected at the 1% significance level, providing strong evidence in favor of national exuberance in the Brazilian commercial real estate market, as indicated by both real prices and the price-to-rent ratio series (as presented in Table 5).Additionally, the date-stamping results presented in Figure 3 clearly demonstrate a distinct period of exuberance in the price-to-rent ratio series[18].

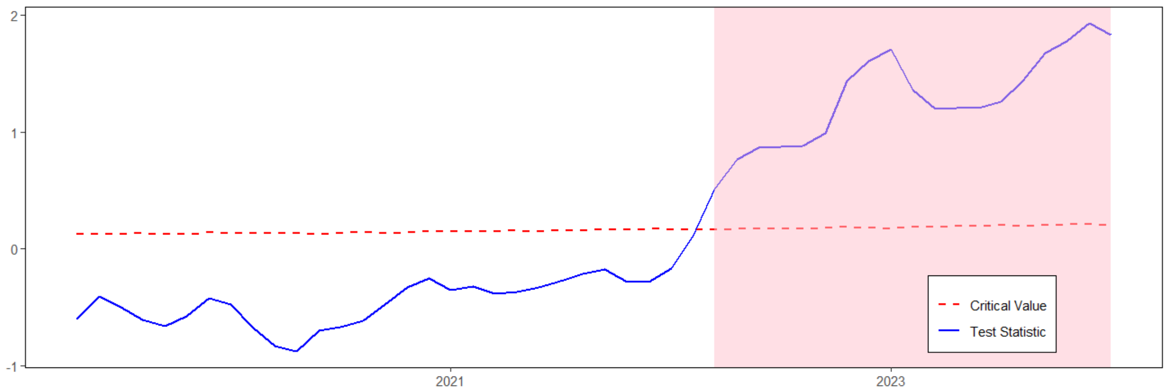

Figure 3: Panel BSADF Statistic and critical value at 90% confidence level.

The panel BSADF statistic sequence remains above its critical value at the 90% confidence level from the beginning of March 2022 until the end of the sample period in October 2023. We find consistency with the results obtained by the univariate GSADF test, indicating a period of bubble deflation towards the end of our sample period. This evidence supports the argument of a possible common factor that induced this dynamic in prices and rents, a movement that once again suggests an increase in the profitability of commercial real estate assets.

4.5 Intrinsic Bubble Tests

To investigate the presence of intrinsic bubbles, we employ the method outlined by Froot and Obstfeld (1991) to estimate all the parameters of our intrinsic bubble test using maximum likelihood[19]. The estimation process was conducted following the approach suggested by Startz (2005), which incorporates an $AR(1)$ process for the

[17] The results presented in this section correspond to the period between January 2018 and October 2023 for all cities in the sample.

[18] The results presented in this section correspond to the period between January 2018 and October 2023 for all cities in the sample.

[19] The maximum likelihood estimation, as suggested by Froot and Obstfeld (1991), allows us to estimate all parameters simultaneously for the intrinsic bubble test. Consequently, it is not necessary to assume any value for the discount factor, as would be the case if we estimated it using OLS.

errors. Table 6 shows the estimated parameters for the intrinsic bubble test equation (29) as outlined in Section 2.7.

Table 6: Results for Intrinsic bubble tests.

	Bubble	c_0	c_1	λ
Panel A: data for the Brazil				
Brazil	No	219.016***	0.000	2.000***
		(11.916)	(675.196)	(0.000)
Panel B: data for the cities				
São Paulo	No	216.366***	0.000	2.000***
		(11.916)	(755.779)	(0.000)
Rio de Janeiro	No	226.527***	0.000	1.000
		(11.916)	(25.340)	(21.794)
Belo Horizonte	No	230.121***	0.000	1.000
		(10.909)	(24.881)	(21.794)
Porto Alegre	No	201.974***	0.004	4.246
		(9.747)	(1, 124, 265)	(1.546, 281)
Brasília	No	132.072***	4.641761	1.784709
		(8.367)	(146.146)	(2480.120)
Campinas	No	78.966***	3.179	2.005
		(8.367)	(297.067)	(3357.807)
Curitiba	No	219.581***	0.011	3.567
		(8.367)	(46883.555)	(1715.533)
Florianópolis	No	219.507***	0.011	3.560
		(8.367)	(98894.920)	(4059.566)
Salvador	No	134.805***	3.640	1.405
		(8.367)	(36.350)	(480.476)

Notes: The table shows the parameters estimates for the intrinsic bubble test. The standard deviation is reported in the parentheses. The model has been estimated through maximum likelihood with error term following a $AR(1)$ process.
*** Indicates statistical significance at the 1% level.
** Indicates statistical significance at the 5% level.
* Indicates statistical significance at the 10% level.

For both the national index and all cities, we cannot reject the null hypothesis of no bubble, as we did not obtain a parameter c_1 significantly different from zero. However, the c_0 representing the fundamental component, as defined in Section 2.2, was statistically significant for all series. This outcome leads us to reject the hypothesis of the presence of a bubble as formulated by Froot and Obstfeld (1991), providing no evidence of an intrinsic bubble driven by movements in rents in a nonlinear fashion in the Brazilian Commercial Real Estate market.

5 Conclusion

This empirical research delves into the examination of the possibility of existence of real estate asset price bubbles in the Brazilian commercial real estate market. To this end, it utilizes monthly time series data over the period from 2012 to 2023, drawing on data from the FIPEZAP research provided by Fipe. The study engages with previous literature by applying five bubble tests originally designed for the stock market to the commercial real estate market.

All methods assume, in some way, that the price, if influenced by a bubble component, will follow a trajectory marked by explosive behavior concerning the fundamental value. Thus, in this research, we use rental prices as a basis for formulating a fundamental value for real prices. Adopting an approach considering a constant discount factor and assuming that the trajectory of log dividends follows a martingale difference, allowed us to assess the presence of explosive behavior in the series through the real prices and price-to-rent ratio.

The initial analysis we conducted was the explosive bubble test, which provided the first evidence supporting the presence of bubbles in the real estate market. This test indicated an explosive behavior across the cities of Porto Alegre, Campinas, and Florianópolis throughout the entire sample period. The result presented by the conventional right tail ADF proved consistent with tests of Engle and Granger (1987) and Johansen (1988), which did not provide evidence in favor of cointegration for these cities.

Delving more deeply into the investigation of the presence of bubbles in prices and extending the analysis proposed by B. Diba and H. Grossman (1984) and Hamilton and Whiteman (1985) in the dynamics of prices and rents, the Periodically Collapsing Bubble test does not provide compelling evidence supporting the presence of a bubble. Nonetheless, the empirical results obtained point towards cointegration for all series and present some evidence of asymmetric convergence regimes for eight out of the ten series. They do not detect a regime in which a sudden rise in stock prices relative to dividends is followed by a crash, whereas a comparable behavior of decreases in stock prices relative to dividends is occasionally observed. Therefore, it does not give us indications in favor of a bubble dynamics, as formulated by Evans (1991)

Furthermore, by employing moving windows, the GSADF test deepened the results obtained from the conventional univariate unit root test. With this test, it was possible to identify periods of bubble inflation and burst, highlighting moments of an unsustainable trajectory of prices diverging from their fundamentals in the cities of São Paulo, Rio de Janeiro, Belo Horizonte, Porto Alegre, Campinas, Florianópolis, Salvador, and in the Brazil index.

Moreover, through a Data Stamping Strategy, the test allowed us to date the periods of exuberance in the series. Through this process, it was possible to identify a bubble formation process in the period of early 2014 until the end of 2018 (evidence found for São Paulo, Rio de Janeiro, and the Brazil index). Additionally, there was a synchronized process of bubble deflation in the period of mid-2021 until the conclusion of the sample period, during which all eight cities experienced a decline in real prices and in the price-to-rent ratio.

In the initial phase, spanning from 2014 to 2018, there was a notable decline in both prices and rents. However, the decrease in rents was more pronounced, leading to a diminished rate of return. A plausible explanation for the formation of this bubble might be associated with a period of economic recession in Brazil characterized by low growth and increased unemployment, coupled with a decline in investment in the construction sector. The second phase attributes the observed changes to the resurgence of commercial activities post-Covid-19, the relaxation of restrictions, and heightened demand due to reduced unemployment and economic growth. This resurgence has revitalized the real estate market, coinciding with a period of increasing interest rates and a subsequent decline in the price-to-rent ratio.

The Panel GSADF test, aligned with the results presented in the univariate GSADF, identifies a period of synchronized exuberance across the cities in our database. This brings further evidence in the direction of a common cause that may have induced this dynamic decline in the price-to-rent ratio, potentially linked to the movement of reopening commercial activities and the resumption of economic activity after the Covid-19 pandemic.

Lastly, the intrinsic bubble test does not reveal any compelling evidence of a nonlinear relationship between fundamentals and prices that could explain excessive price fluctuations during exuberance periods. Therefore, we conclude that there is no empirical evidence of a bubble driven by intrinsic factors such as rent, as formulated by Froot and Obstfeld (1991).

6 Appendix

6.1 A.1. Properties of log Dividends Over Time

When formulating equation (5), we made the assumption that the log-dividend process adheres to a martingale with a trend. In this appendix, we briefly analyze the time-series data related to the dividend-generating process to determine its compatibility with our assumption. Table 7 presents tests for the null hypothesis for log-rents, $d_t = \log(D_t)$.

Table 7: Results from the ADF unit root test for log rents and prices.

	Reject the hypothesis of unit root	ADF no linear trend
Panel A: data for the Brazil		
Brazil	No	-1.2183
Panel B: data for the cities		
São Paulo	No	-1.521
Rio de Janeiro	No	-0.3846
Belo Horizonte	No	-0.357
Porto Alegre	No	-0.681
Brasília	No	-0.1673
Campinas	No	-1.075
Curitiba	No	-1.2763
Florianópolis	No	-1.707
Salvador	No	-1.4424

Notes: The table presents the Augmented Dickey-Fuller with a constant and no linear trend, and with a constant and linear trend for the log real rent series. The second column indicates when we can reject the hypothesis of the presence of a unit root in the log rent series. The lag is selected by Akaike Information Criterion (AIC). *** Indicates statistical significance at the 1% level.
** Indicates statistical significance at the 5% level.
* Indicates statistical significance at the 10% level.

The results is align with the findings reported in Table 1, as presented in Section 4.1. The tests considered were unable to reject the null hypothesis of a unit root for any log-rent series under consideration, and as a result, our findings do not provide sufficient evidence to reject the hypothesis that log dividends follow a martingale.

Additionally, to constitute a valid solution to equation (2), the real price in equation (5) requires that investors' conditional expectation in t of d_{t+1} is equal to $\mu + d_t$. As a result, the disturbance ε_{t+1} in equation (4) must not only be unpredictable based on the past history of dividends but also unpredictable given any broader time information set that investors employ. Specifically, since investors' forecasts of future dividends should rely solely on current dividends, real estate prices (presumed to encapsulate information beyond dividends) should not improve the accuracy of dividend forecasts based solely on current dividends. This assumption is strong, and its validity needs to be investigated based on the data (Froot and Obstfeld 1991).

Thus, as done by Froot and Obstfeld (1991) and A.-S. Chen, L.-Y. Cheng, and K.-F. Cheng (2009), we explored the Granger (1969) causality test to analyse prices and rents, enabling us to assess the ability to predict future rents in a time series using prior values of prices, and reciprocally.

The table 8 reports tests for Granger-causality between log real prices and log real rents. In each row, an F

Table 8: Results from the Granger causality test for log of rents and prices.

Variable	F test	
	$\Delta p_t = \alpha_1 + \sum_{i=1}^{m} \beta_{1,i} \Delta d_{t-i} +$ $+ \sum_{i=1}^{m} \gamma_{1,i} \Delta p_{t-i} + \epsilon_{1,t}$	$\Delta d_t = \alpha_2 + \sum_{i=1}^{m} \beta_{2,i} \Delta d_{t-i} +$ $+ \sum_{i=1}^{m} \gamma_{2,i} \Delta p_{t-i} + \epsilon_{2,t}$
Panel A: data for the Brazil		
Brazil	1.5779	0.2919
Panel B: data for the cities		
São Paulo	0.9724	0.8032
Rio de Janeiro	2.0434	2.3191
Belo Horizonte	7.9399***	0.6973
Porto Alegre	1.0224	0.1087
Brasília	1.6146	1.6567
Campinas	1.8656	0.3272
Curitiba	2.206	1.161
Florianópolis	0.5223	1.9819
Salvador	2.206*	1.161

Notes: The table presents the Granger Causality test with lag $m = 1$ for the real prices rent series. The second column indicate if we have evidence of rents Granger-cause prices. Different lag lengths were experimented with in these regressions; however, the results remained unchanged.
*** Indicates statistical significance at the 1% level.
** Indicates statistical significance at the 5% level.
* Indicates statistical significance at the 10% level.

test is presented for the hypothesis null that these coefficients are jointly zero. The results of the test suggest that we cannot reject the hypothesis that p_t, has no incremental power for forecasting future dividend changes in dividends for all series considered. Conversely, there is some evidence for the Salvador and Belo Horizonte series that rents Granger-cause prices.

6.2 A.2. Data Stampling Strategie for real prices

Considering the results in Section 4.3, we can present the outcomes of the date stamping strategy for real prices as follows:

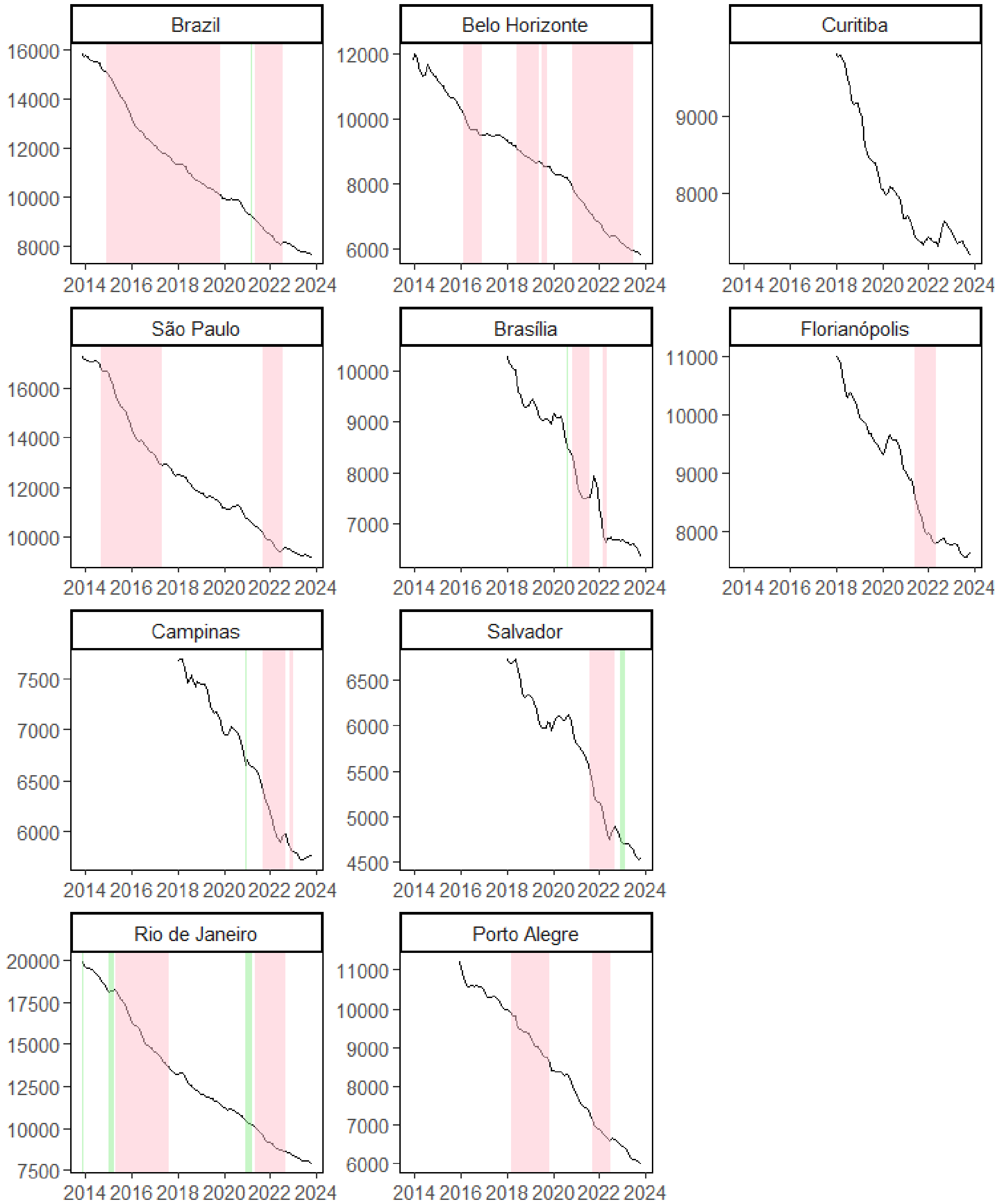

Figure 4: Graphical representation of episodes of exuberance in real prices for the Brazil index and nine Brazilian cities.

In each graph in the Figure 4, the shaded area represents the periods of exuberance detected by the methodology described in Section 2.5. The green area signifies the periods of exuberance where ΔP_t is positive, indicating a growth period in the prices. Conversely, the red shaded area represents the periods of degrowth in the prices, corresponding to the periods of exuberance where ΔP_t is negative.

In Figure 5, we observe a graphical representation of the critical values at a 10% significance level and the

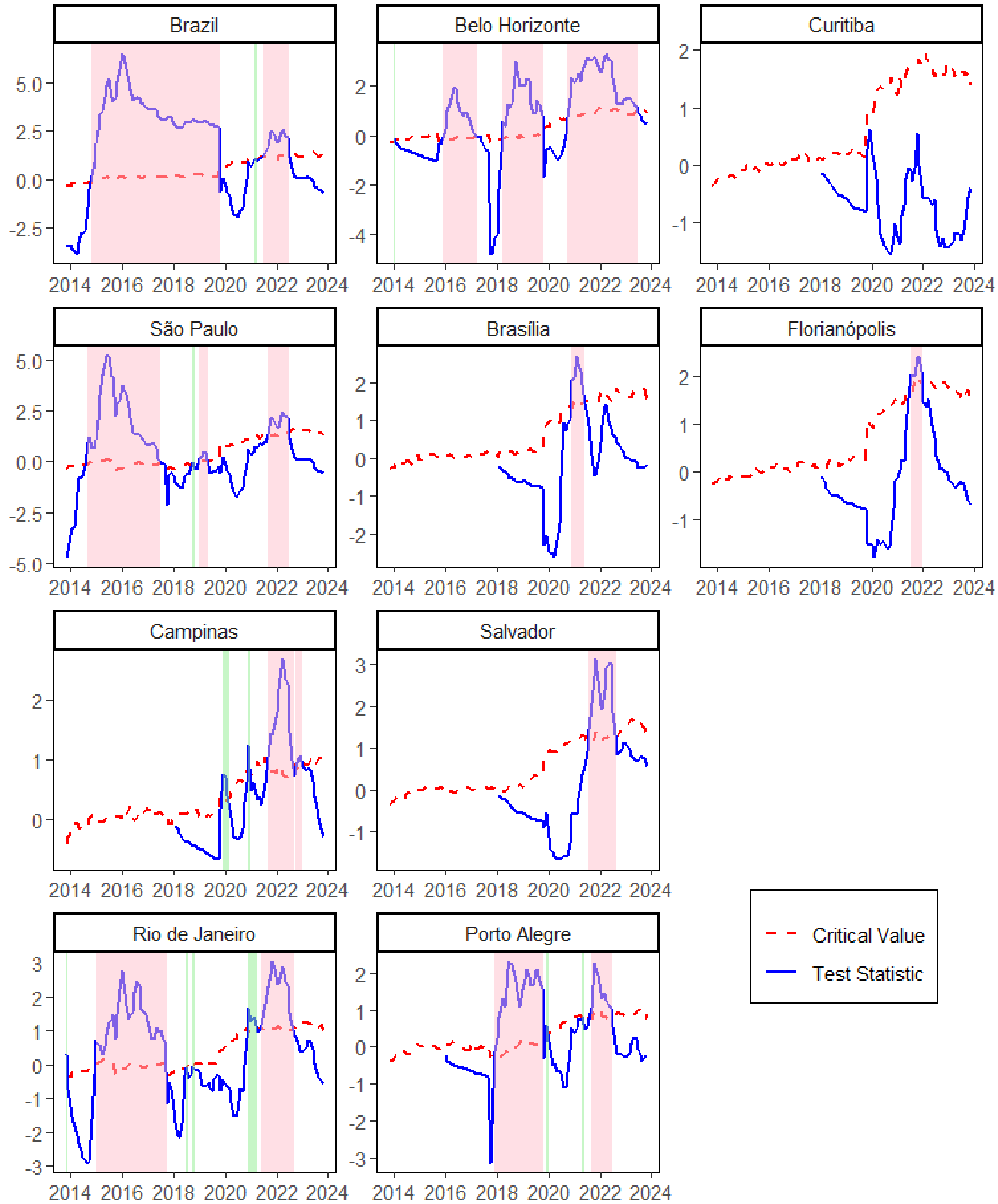

Figure 5: Test statistic and critical values (at a 10% significance level) for real prices in the Brazil index and nine Brazilian cities.

test statistic (defined by the equation (24) in the section 2.5) for real prices enables us to estimate the origination and termination dates of exuberance periods.

As there is a decline in real prices for all price ranges for the period under consideration, all identified periods of exuberance indicate a rapid fall in prices. However, they cannot be interpreted as an explosive bubble process without first considering the effect of fundamentals on prices, as presented in section 2.2. Therefore, the analysis will be complemented with Appendix 6.3 using the price-to-rent series as present in section 4.3.

6.3 A.3. Data Stampling Strategie for price-to-rent ratio

Similar to the Appendix 6.2, we can also present the outcomes of the date stamping strategy price-to-rent ratio as follows:

Figure 6: Graphical representation of episodes of exuberance in the price-to-rent ratio for the Brazil index and nine Brazilian cities.

In each graph in the Figure 6, the shaded area represents the periods of exuberance detected by the methodology described in Section 2.5. The green area signifies the periods of exuberance where $\Delta\left(P_t/D_t\right)$ is positive, indicating the inflationary bubble period. Conversely, the red shaded area represents the periods of deflation in the bubble, corresponding to the periods of exuberance where $\Delta\left(P_t/D_t\right)$ is negative.

In Figure 7, we observe a graphical representation of the critical values at a 10% significance level and the

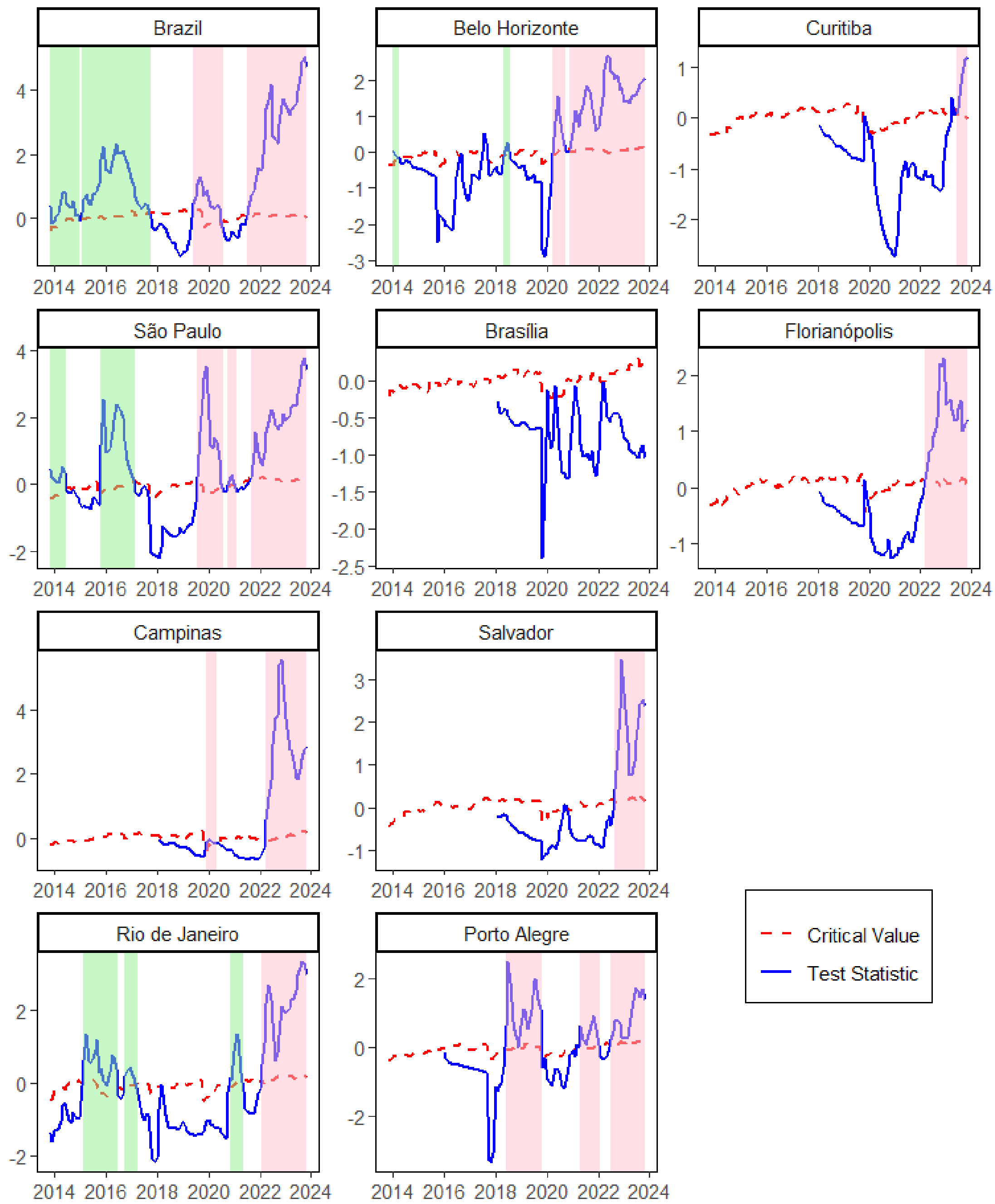

Figure 7: Test statistic and critical values (at a 10% significance level) in the price-to-rent ratio for the Brazil index and nine Brazilian cities.

test statistic (defined by the equation (24) in the section 2.5) for the difference between price and fundamental enables us to estimate the origination and termination dates of exuberance periods.

Through these results, a trend of increase in the price-to-rent ratio, consequently indicating profitability in commercial real estate assets, becomes apparent. This upward movement, highlighted in green above, can be traced from early 2014 until the end of 2018 in the series for São Paulo, Rio de Janeiro, Belo Horizonte, and São Paulo.

Moreover, during the period between mid-2018 and the beginning of 2020, a downturn is evident for the cities of Porto Alegre, Campinas, and São Paulo. Finally, another declining trend can be observed from mid-2021 until the conclusion of the sample period. However, this time the evidence is present in seven cities, namely

São Paulo, Rio de Janeiro, Belo Horizonte, Campinas, Florianopolis, Salvador, and Porto Alegre.

In this manner, as presented in section 4.3, the initial exuberance period serves as strong evidence for a bubble formation movement, as formulated by P. C. B. Phillips, Shi, and Yu (2015a) and Efthymios Pavlidis, Yusupova, et al. (2016). Furthermore, the subsequent two periods mentioned above can be interpreted as a process of deflation of this bubble.

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