Department of Economics- FEA/USP

Optimal patent breadth in a horizontal innovation growth model

REINAN RIBEIRO, DAVID TURCHICK

WORKING PAPER SERIES Nº 2014-15

DEPARTMENT OF ECONOMICS, FEA-USP
WORKING PAPER Nº 2014-15

Optimal patent breadth in a horizontal innovation growth model

Reinan Ribeiro (reinan.ribeiro@fgvmail.br)

David Turchick (dturchick@usp.br)

Abstract:

Patenting and licensing are introduced into the lab-equipment horizontal innovation model of endogenous growth. Patent length is not deterministic, in that patent holders' market power will collapse if and only if their patented goods get successfully imitated. The model is solved in an exact fashion. Consideration of static and dynamic inefficiencies shows that optimal intellectual property rights protection entails maximal patent length, but nonmaximal patent breadth. By fixing a lower patent breadth level, government is able to trade off a modest negative long-term effect on growth for a substantial positive short-term effect on consumption. A third potential source of inefficiency, reputational, is also considered.

Keywords: intellectual property rights, horizontal innovation, lab-equipment model, growth, patent length, patent breadth.

JEL Codes: L13, O31, O34, O41.

Optimal patent breadth in a horizontal innovation growth model

Reinan Ribeiro*, David Turchick†

June 2014

Abstract

Patenting and licensing are introduced into the lab-equipment horizontal innovation model of endogenous growth. Patent length is not deterministic, in that patent holders' market power will collapse if and only if their patented goods get successfully imitated. The model is solved in an exact fashion. Consideration of static and dynamic inefficiencies shows that optimal intellectual property rights protection entails maximal patent length, but nonmaximal patent breadth. By fixing a lower patent breadth level, government is able to trade off a modest negative long-term effect on growth for a substantial positive short-term effect on consumption. A third potential source of inefficiency, reputational, is also considered.

Keywords: intellectual property rights; horizontal innovation; lab-equipment model; growth; patent length; patent breadth

JEL codes: L13; O31; O34; O41

*Reinan Ribeiro is a researcher at the Rio School of Economics of the Getulio Vargas Foundation (EPGE-FGV). E-mail address: reinan.ribeiro@fgvmail.br.

†Corresponding author. David Turchick is an Assistant Professor at the Department of Economics of the University of São Paulo (FEA-USP). Telephone number: +55 (11) 3091-5802. E-mail address: dturchick@usp.br.

1 Introduction

The issue of protection of intellectual property rights (IPR) through patents presents the policymaker with a tradeoff. This can be understood as one of comparing the inefficiency arising from market power with that arising from insufficient incentives for technical progress.

Part of the literature on IPR protection (including, for instance, Tandon, 1982; Judd, 1985; and Gilbert and Shapiro, 1990) concludes that an infinite patent length is socially optimal. A more recent and larger portion of this broad literature (see, for instance, Horowitz and Lai, 1996; Koléda; 2004; Horii and Iwaisako, 2007; Futagami and Iwaisako, 2007; Furukawa, 2007; Bessen and Maskin, 2009; and Chen and Iyigun, 2011), though, makes the case for finite-term patents.

Nevertheless, two papers in this more recent strand of the literature (Kwan and Lai, 2003; Cysne and Turchick, 2012), using a horizontal innovation R&D model, point back in the direction of the results obtained in the 1980's and 1990. There, a "lab-equipment" endogenous growth model (Rivera-Batiz and Romer, 1991) coupled with an exogenous imitation rate is solved in an analytical, exact fashion. That solution is applied to study the policy tradeoff between weaker levels of IPR protection (which boost consumption in the short run) and stronger levels (which aid growth). The latter of these two papers shows that, under logarithmic utility, this tradeoff has a "corner solution", in the sense that the government should necessarily choose to prescribe and seek to enforce an infinite patent length. That is, the "static efficiency" / "short term" / "level of GDP" view in the optimal IPR protection policy debate (more protection resulting in greater monopoly rents and deadweight losses and lower current production and consumption levels) would be dominated by the opposing "dynamic efficiency" / "long term" / "growth of GDP" view (more protection boosting technology-based growth and thus yielding higher future consumption levels).

In these works, a simplification proposed in Romer (1990) is in order: the R&D sector (which produces designs for new capital goods) and the intermediate sector (which employs these designs to produce capital goods) can be amalgamated. That is, the inventor and the producer of a specific capital good can be interpreted as the same firm. However, in these works, the only instrument government has in order to curtail monopoly power is the adjustment of patent length (or some policy variable assumed to move one-to-one with length).

The present work recognizes that the issue of optimal IPR protection is not unidimensional. Here, as in Rivera-Batiz and Romer (1991), R&D firms seek to protect their intellectual property through patents, and license these patents to the producers of intermediate goods, who belong to a different sector of the economy. IPR protection can be affected not only through patent length regulation, but also patent breadth. Licensing fees provide a simple and direct way to measure patent breadth. In fact, when breadth is set at a higher (lower) level, inventors' fear of erosion of their monopoly power through legally successful imitation is reduced (increased), and are therefore able to extract more (less) rent, charging a higher (lower) fee.

Our baseline scenario is one in which the government is able to adjust the level of patent breadth, alongside with patent length, through specific IPR protection policy. Other works addressing the issue of the optimal level of patent breadth (in different growth models) are Gilbert and Shapiro (1990), Klemperer (1990), Li (2001), Goh and Olivier (2002), Furukawa (2007) and Chu and Furukawa (2011). Here it will be shown that, although the same short-run-vs.-long-run tradeoff mentioned for patent lengths is also present in the choice of optimal breadths, there is not a single view that should be prevalent in the determination of the optimal mix of IPR protection policies. While the dynamic efficiency view should still be dominant in the choice of the level of patent length (even if there is room to raise IPR protection in terms of breadth instead), both the dynamic and the static efficiency views will be important in the determination of patent breadth. By lowering patent breadths, government is able to trade off a modest negative long-term effect on growth for a substantial positive short-term effect on consumption.

An alternative scenario would be one in which setting patent length or breadth to some specific level is beyond the capacity of the policymaker, and is actually done by judges and justices, given their level of discretion and/or whatever constraints entering their decision problem, such as jurisprudence. In either case, it will be shown that, by enacting specific regulations in the R&D and in the capital goods sectors, the decentralized economy can generate a socially optimal allocation.

A delicate issue in this type of model is that of time consistency. Once a high level of IPR protection is granted, a myopic government may be tempted to break patents, thus collecting not only the long-run benefits, but also the short-run ones. In order to treat this issue, we also analyze equilibria where the optimal mix of IPR protection policies is invariant with respect to the current mix. As explained above, this will actually be a restriction on the model's set of equilibria only along the breadth dimension.

The model is introduced in the next section. In section 3 we solve (in an exact fashion, no linearizations involved) for the equilibrium path, and verify that the present model allows for a rigorous presentation of the IPR protection tradeoff faced by the policymaker with respect to patent breadth. In section 4, the exact solution found in the previous section will be applied in order to discuss optimal levels for patent length and breadth. Section 5 deals with the intertemporal consistency issue of IPR protection policies, while section 6 deals with the issue of social optimality of the decentralized equilibrium. Section 7 concludes.

2 The model

The present R&D endogenous growth model is based on Kwan and Lai (2003) and Gancia and Zilibotti (2005, section 2.3). The main addition to their story will be that innovators/inventors may choose to license their patents.

Households maximize

$$U = \int_0^\infty e^{-\rho t} \log c_t dt$$

subject to $\dot{b}_t = w_t + r_t b_t - c_t$, where consumption c is in terms of the final good, w is the wage (labor is inelastically supplied to the final-good sector), r is the rate of return on assets held b, given b_0 and the usual no-Ponzi game condition. This yields the standard Euler equation

$$\gamma := \hat{c} = r - \rho \tag{1}$$

(throughout the paper, "ˆ" will stand for growth rate), as well as the transversality condition $r > \gamma$ (innocuous since $\rho > 0$).

Firm $i \in I$ produces final goods according to the homogeneous production function

$$Y_i = L_i^{1-\alpha} \int_0^A x_{i,j}^\alpha dj,$$

where L_i is labor input (inelastically supplied and constant over time), $x_{i,j}$ is the quantity of index-j intermediate good being used as input, $\alpha \in (0,1)$ and A is the measure of existing intermediate goods. In order to maximize profit $1Y_i - wL_i - \int_0^A p_j x_{i,j} dj$, the demand of firm i for capital good j satisfies

$$x_{i,j} = L_i \left(\frac{\alpha}{p_j}\right)^{\frac{1}{1-\alpha}}.$$

Let

$$x_j := \sum_{i \in I} x_{i,j} = L \left(\frac{\alpha}{p_j}\right)^{\frac{1}{1-\alpha}}, \tag{2}$$

where $L := \sum_{i \in I} L_i$, be aggregate demand for j. Note that $x_{i,j}/L_i = x_j/L$, so

$$Y := \sum_{i \in I} Y_i = \sum_{i \in I} L_i^{1-\alpha} \int_0^A \left(\frac{L_i x_j}{L}\right)^\alpha dj = \sum_{i \in I} L_i \int_0^A \left(\frac{x_j}{L}\right)^\alpha dj = L^{1-\alpha} \int_0^A x_j^\alpha dj.$$

In order to protect its intellectual property (and subsequent profit), firm j (the inventor of intermediate j) has a choice between relying on industrial secrecy or applying for a patent. As stressed in Lemley and Shapiro (2005), patents are of a probabilistic nature. They only make it possible for j to sue a competing company who seems to have copied its invention; they do not guarantee j will win the suit. And once they lose, although the patent is still there, it becomes useless since j's market power has eroded anyway. It thus makes sense thinking of real, effective patent lengths as random variables, even though law prescribes a nominal, fixed duration.[1]

Let $[0, A_c]$ be the set indexing capital goods which have been successfully copied (meaning perfect substitutes are already being legally produced and marketed), while $(A_c, A]$ are those still being effectively protected. Following Krugman (1979), we assume that the measure A_c follows

$$\dot{A}_c = m\left(A - A_c\right), \tag{3}$$

where $m \geq 0$ is the imitation/patent-termination rate, and $A(0)$ and $A_c(0)$ are given. Patent termination only occurs when one of the $A - A_c$ currently patented goods gets succesfully copied.

It may be noted that a patent term T can be associated with m in a natural way. If $m = 0$, no imitation ever occurs, and $T = \infty$. If $m > 0$, assuming that patent termination time follows an exponential distribution with parameter m (patents last m^{-1} on average) and that innovators are risk neutral, they will be indifferent between the certain profit stream $\int_0^T \pi e^{-rs} ds = (\pi/r)\left(1 - e^{-rT}\right)$ and the average present value of their future profits $\int_0^\infty \left(\int_0^t \pi e^{-rs} ds\right) m e^{-mt} dt = (\pi/r) \int_0^\infty \left(1 - e^{-rt}\right) m e^{-mt} dt = (\pi/r)\left(1 - m/\left(r + m\right)\right) = \pi/\left(r + m\right)$. Equating these two expressions gives[2]

$$m = \frac{r}{e^{rT} - 1}. \tag{4}$$

[1] For a similar model with deterministic patent terms, see Iwaisako and Futagami (2003).

[2] Expression 4 is equivalent to equation 30 in Kwan and Lai (2003).

Because the imitation rate m is exogenous to the innovator, and does not depend on his decision between relying on secrecy or applying for a patent and therefore disclosing its invention (see Gallini, 2002; and Erkal, 2005, for a discussion and treatment of this possibility), and since there are no costs such as maintenance fees for patents, in equilibrium the innovator will necessarily apply for a patent. Upon its granting, it may be licensed to other firms who wish to produce that capital good. The licensees must pay to the licensor a royalty fee $\delta > 0$ per unit sold of that good during the patent lifetime. This will be our measure of patent breadth, since more (less) breadth is necessarily associated with a higher (lower) probability that the inventor will effectively maintain his monopoly power during the patent lifetime, and hence with the expectation of extracting more (less) monopoly rents and the charging of a higher (lower) δ.

For the time being, we assume that patents' breadth level belongs to the policymaker's toolkit, and can effectively be achieved through patent legislation. It could be argued that, in reality, to a large extent, patent breadth is determined directly by judges who analyze suits of violation of IPR, based on the available jurisprudence and/or on their personal views of fairness (or other personal inclinations), which cannot be altered at the policymaker's will. This interpretation will be dealt with in section 6.

The market for capital goods is monopolistically competitive. Firms producing intermediate j face the marginal cost $1 + \delta$, since they must purchase one unit of final good (at a price normalized at 1) in order to produce one unit of intermediate good, besides paying the royalty fee δ to firm j (the inventor of intermediate j). Since these capital goods enter the homogeneous production function of the final goods sector in a symmetric fashion, firm j must charge the price $p_j = 1 + \delta$ during the lifetime of the patent, and $p_j = 1$ from that point on.[3] Thus, the quantity demanded of intermediate j in equilibrium, according to (2),

[3] Becker and Lu (2010) use royalty rate data collected from RoyaltySource and show its positive correlation with markup, so that royalty rates (whenever the information is available and believed to be reliable) can also be used as a measure of market concentration.

is[4]

$$x_j = \begin{cases} x_c = L\alpha^{\frac{1}{1-\alpha}}, & \text{if } j \in [0, A_c] \\ x_m = L\left(\frac{\alpha}{1+\delta}\right)^{\frac{1}{1-\alpha}}, & \text{if } j \in (A_c, A] \end{cases} . \tag{5}$$

Since there is free entry into the R&D sector, the present value of the returns from inventing capital good j must equal the cost of invention $\beta > 0$. That is, if $m = 0$, $\beta = \int_0^\infty \pi e^{-rs} ds = \pi/r$, and if $m > 0$, $\beta = \int_0^\infty \left(\int_0^t \pi e^{-rs} ds\right) m e^{-mt} dt = \pi/\left(r+m\right)$. Thus, in either case, since $\pi = \delta x_m$, we get

$$\beta = L\frac{\delta}{r+m}\left(\frac{\alpha}{1+\delta}\right)^{\frac{1}{1-\alpha}}, \tag{6}$$

whence

$$r = \frac{L}{\beta}\delta\left(\frac{\alpha}{1+\delta}\right)^{\frac{1}{1-\alpha}} - m. \tag{7}$$

Therefore this economy presents no transitional dynamics for the interest rate r, which immediately responds to changes in m and/or δ so that the arbitrage relation (7) remains valid. Since the value of inventions, $\pi/\left(r+m\right)$, as a function of δ increases up to $\delta = 1/\alpha - 1$ and decreases thereafter (see (6)), a $\delta > 1/\alpha - 1$ policy would never be binding, in the sense that inventors themselves would prefer to license their inventions for less. Thus, we restrict attention to the $[0, 1/\alpha - 1]$ interval for the policy variable δ.[5]

Still because $\delta\left(\alpha/\left(1+\delta\right)\right)^{\frac{1}{1-\alpha}}$ increases with δ in this interval, equation (7) shows that greater incentives to potential innovators in terms of a stricter patent protection regulation,

[4] It should be noted that at the moment j is invented, say at time t_1, it equals $A\left(t_1\right)$. During the lifetime of its patent it still belongs to the $\left(A_c\left(t\right), A\left(t\right)\right]$ interval (the *second* line in (5)), corresponding to the lower, monopoly production x_m. Once its patent becomes ineffective (assuming $m > 0$), say at time t_2, then $j = A_c\left(t_2\right)$, and from that point on, we have $j \in \left[0, A_c\left(t\right)\right]$ (the *first* line in (5)), corresponding to the higher, competitive production x_c.

[5] This is in agreement with the model in Kwan and Lai (2003) and Cysne and Turchick (2012), where maximum IPR protection in terms of patent breadths is implicitly assumed: $\delta = 1/\alpha - 1 \Rightarrow p_j = 1/\alpha$ and $x_m = L\alpha^{\frac{2}{1-\alpha}}$.

whether in terms of breadth (a larger δ) or length (a lower m), lead to a higher equilibrium interest rate, in order to match their proceeds.

Equations (4) and (7) taken together confirm that the patent termination rate m is negatively correlated with the measure of patent length T. Just rewrite (4) as

$$T = \frac{1}{r}\log\left(1+\frac{r}{m}\right) = \frac{1}{r+m-m}\log\frac{r+m}{m}, \tag{8}$$

and since $r+m$ has no dependence on m from (7),

$$\frac{\partial T}{\partial m} = \frac{1}{r^2}\log\frac{r+m}{m} + \frac{1}{r}\frac{m}{r+m}\left(-\frac{r+m}{m^2}\right) = \frac{1}{r^2}\left(\log\left(1+\frac{r}{m}\right)-\frac{r}{m}\right) < 0.$$

The parameter m can thus be legitimately interpreted as an inverse measure of IPR protection in terms of patent length (not only the inverse of the first moment of the distribution of lengths). We may refer to it as the "inverse-length parameter".

The resource constraint of the economy reads

$$C + \beta\dot{A} = Y - X, \tag{9}$$

where $C := Lc$ is aggregate consumption, $Y = L^{1-\alpha}\int_0^A x_j^\alpha dj = L^{1-\alpha}\left(A_c x_c^\alpha + (A-A_c)\,x_m^\alpha\right)$ is total output and $X := \int_0^A x_j dj = A_c x_c + (A-A_c)\,x_m$ is total intermediate goods production.

It may be noted that, substituting (5) in these expressions, one gets $X = L\left(A_c\alpha^{\frac{1}{1-\alpha}} + (A-A_c)\left(\alpha/\left(1+\delta\right)\right)^{\frac{1}{1-\alpha}}\right) \leq L\left(A_c\alpha^{\frac{1}{1-\alpha}} + (A-A_c)\,\alpha^{\frac{1}{1-\alpha}}/\left(1+\delta\right)^{\frac{\alpha}{1-\alpha}}\right) = \alpha Y$, where equality holds only if inventors have no monopoly power at all, that is, $\delta = 0$.

3 Equilibrium

From (9) and (5) one gets

$$\hat{A} = \frac{L}{\beta}\left[\frac{A_c}{A}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) + \frac{A - A_c}{A}\left(\left(\frac{\alpha}{1+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha}{1+\delta}\right)^{\frac{1}{1-\alpha}}\right)\right] - \frac{C}{\beta A}.$$

Following Kwan and Lai (2003), consider two new variables, a scaled version of consumption, $h := C/(\beta A)$, and the fraction of intermediate goods that have already lost their patents, $g := A_c/A$. Differentiating h and g with respect to time and plugging in the expression for $\hat{A}$ yields the model's equilibrium dynamics:

$$\begin{cases} \dot{g} = m + (\kappa_2 - m)\, g + \kappa_1 g^2 + gh \\ \dot{h} = (\gamma + \kappa_2)\, h + \kappa_1 gh + h^2 \end{cases}, \tag{10}$$

where $\kappa_1 := \left(1 - \alpha + \delta - (1-\alpha)(1+\delta)^{\frac{1}{1-\alpha}}\right)(r+m)/(\alpha\delta)$ and $\kappa_2 := -(1-\alpha+\delta)(r+m)/(\alpha\delta)$ (where $r + m$ comes from (7)). It may be noted that both κ_1 and κ_2 are negative.[6]

If $m + \gamma > 0$, the only possible steady state (corresponding to a balanced growth path in the original variables of the model) with a positive consumption level (this will be checked below) is

$$\left(\overline{g}, \overline{h}\right) = \left(\frac{m}{m+\gamma}, -\kappa_1 \frac{m}{m+\gamma} - \gamma - \kappa_2\right). \tag{11}$$

In case $m + \gamma = 0$, since $m \geq 0$ and $\gamma \geq 0$, we have $m = \gamma = 0$, whence (10) leads to a continuum of (non-isolated) steady states of the form $\left(\overline{g}, \overline{h}\right) = (x, -\kappa_1 x - \kappa_2), \forall x \in [0,1]$.

The assumption of a steady state for h implies that $\hat{A}$ is asymptotically equal to γ. Hence we impose that the parameters are such that r is at least as large as ρ, whence plugging (7)

[6] For κ_2 this is clear. As for κ_1, Bernoulli's inequality gives $(1+\delta)^{\frac{1}{1-\alpha}} > 1 + \delta/(1-\alpha)$, whence $0 < -(1-\alpha+\delta) + (1-\alpha)(1+\delta)^{\frac{1}{1-\alpha}} = -\alpha\delta\kappa_1/(r+m)$, and κ_1 is negative indeed.

into (1) yields a nonnegative growth rate. For the reader's convenience, we collect this and all the aforementioned constraints into the feasibility set

$$F := \left\{ (m, \delta) \in \mathbb{R}_+ \times \left[\underline{\delta}, \frac{1}{\alpha} - 1 \right] : m \leq \frac{L}{\beta} \delta \left(\frac{\alpha}{1+\delta} \right)^{\frac{1}{1-\alpha}} - \rho \right\},$$

where $\underline{\delta} \in (0, 1/\alpha - 1)$ is such that $(L/\beta)\, \underline{\delta} \left(\alpha / \left(1 + \underline{\delta}\right)\right)^{\frac{1}{1-\alpha}} = \rho$. Since $(L/\beta)\, \delta \left(\alpha / \left(1 + \delta\right)\right)^{\frac{1}{1-\alpha}}$ is continuous in δ, this $\underline{\delta}$ exists if we additionally assume (and we do so) $\rho < (L/\beta)\,(1/\alpha - 1)\, \alpha^{\frac{2}{1-\alpha}}$. And since it is a strictly increasing function of δ, $\underline{\delta}$ is well defined, so that lower values of δ cannot take place in F, while larger values can (consider pairs $(0, \delta)$ for instance).

Again from (7) one can see that the restriction imposed within F is simply $m \leq r + m - \rho$, or $\rho \leq r$. Thus (1) yields a nonnegative growth rate $\gamma = r - \rho \geq 0$ indeed. Transversality holds automatically from $\rho > 0$ and (1). As for the validity of the $\overline{h} > 0$ condition within F, note that $\overline{h}$ increases with m (because $-\kappa_1$ is positive; because (7) shows that κ_1 and κ_2 do not depend on m since $r + m$ does not either; and because the same equation shows that both $m/\left(m + \gamma\right) = m/\left(r + m - \rho\right)$ and $-\gamma = \rho - (r + m) + m$ increase with m). Therefore it is sufficient to consider the $m = 0$ case, for which $\overline{h} = -\gamma - \kappa_2 = \rho - (r + m) - \kappa_2 = \rho + \left((1 - \alpha + \delta) / (\alpha\delta) - 1\right)(r + m) = \rho + (1 - \alpha)(1 + \delta)(r + m) / (\alpha\delta) > 0$.

The system of differential equations (10) is one of the (matricial) Riccati type, and can be solved in an exact fashion by the method proposed in Levin (1959, Theorem 2), according to which solutions to (10) can be found via solutions to the auxiliary system of linear differential equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & m \\ 0 & m + \gamma & 0 \\ -\kappa_1 & -1 & m - \kappa_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We are only interested in the saddle path, which will take the form

$$\begin{bmatrix} g(t) \\ h(t) \end{bmatrix} = \begin{bmatrix} \zeta \\ 0 \end{bmatrix} + \frac{1}{1-\frac{g_0-\overline{g}}{g_0-\zeta}e^{-(\gamma+\kappa_2+\kappa_1\zeta)t}} \begin{bmatrix} \overline{g}-\zeta \\ \overline{h} \end{bmatrix}, \tag{12}$$

where g_0 is the given $g(0)$ (since both $A(0)$ and $A_c(0)$ are given), and $\zeta := \left(m - \kappa_2 + \sqrt{(m-\kappa_2)^2 - 4\kappa_1 m}\right) / (2\kappa_1)$ is the negative root of $\kappa_1 x^2 + (\kappa_2 - m)x + m = 0$. The $m + \gamma = 0$ case leads to trivial (degenerate) positive consumption equilibrium "paths": $(g(t), h(t)) = (g_0, -\kappa_1 g_0 - \kappa_2) = (\overline{g}, \overline{h})$.

Since $h(t)/\overline{h} = (g(t) - \zeta)/(\overline{g} - \zeta)$, we can solve for h_0 $(= h(0))$:

$$h_0 = \frac{g_0 - \zeta}{\overline{g} - \zeta}\overline{h}. \tag{13}$$

Therefore changes in m affect h_0 through their impact on ζ and $\gamma = r + m - m - \rho$ (with $r + m$ given by (7)), while δ affects h_0 through those same channels, plus κ_1 and κ_2.

The positivity of h_0 derives directly from that of $\overline{h}$ within F. An alternative expression that will be useful in Proposition 1 below is

$$h_0 = (\kappa_1\zeta - m - \gamma)\left(1 - \frac{g_0}{\zeta}\right). \tag{14}$$

In fact, the expression for the steady state (11) and the identity $\kappa_1\zeta^2 + (\kappa_2 - m)\zeta + m = 0$ give

$$(\overline{g} - \zeta)(m + \gamma - \kappa_1\zeta) = m + (-\kappa_1\overline{g} - m - \gamma)\zeta + \kappa_1\zeta^2 = (-\kappa_1\overline{g} - \gamma - \kappa_2)\zeta = \overline{h}\zeta,$$

whence (13) yields (14).

It can be shown that the lab-equipment model equips us with a rigorous explanation of the well-established intuition that the issue of IPR protection in terms of patent length presents policymakers with a tradeoff: too low hinders growth, too high cuts immediate consumption.[7] The next proposition shows that the present model yields this same IPR protection tradeoff in terms of breadth. In other words, say government has decided to strengthen patent protection only in terms of breadth, from δ to $\delta' > \delta$. Then the growth rate of consumption γ (which is also the long-run growth rate of the economy) will be raised to $\gamma' > \gamma$, but there will be an immediate downward jump in consumption h, from $h_0 = \overline{h}$ to $h_0' < \overline{h}$.

Proposition 1 *The pursuance of stronger IPR protection through an increase in the patent breadth level brings, cœteris paribus, (i) faster long-run growth of the economy ("dynamic efficiency gains"), and (ii) an immediate negative effect on consumption ("static efficiency losses").*

Proof. Take $(m, \delta) \in F$ with $\delta < 1/\alpha - 1$. For (i), we must show that $\partial\gamma/\partial\delta > 0$. Expression 7 gives

$$\begin{aligned}\frac{\partial r}{\partial \delta} &= \frac{L}{\beta}\alpha^{\frac{1}{1-\alpha}}\left((1+\delta)^{-\frac{1}{1-\alpha}} - \frac{1}{1-\alpha}\delta\,(1+\delta)^{-\frac{1}{1-\alpha}-1}\right) \\ &= \frac{L}{\beta}\left(\frac{\alpha}{1+\delta}\right)^{\frac{1}{1-\alpha}}\left(1 - \frac{1}{1-\alpha}\frac{\delta}{1+\delta}\right) = \frac{1-\alpha\,(1+\delta)}{\delta}\frac{r+m}{(1-\alpha)\,(1+\delta)}. \end{aligned} \tag{15}$$

From the Euler equation (1), we see that $\partial\gamma/\partial\delta = \partial r/\partial\delta$, which is positive.

For (ii), we must prove $\partial h_0/\partial\delta < 0$. In order to derive (14) with respect to δ, we must perform a few auxiliary calculations. First note that $\kappa_1 + \kappa_2 = -(1-\alpha)\,(1+\delta)^{\frac{1}{1-\alpha}}\,(r+m)\,/\,(\alpha\delta)$ which, from (7), equals $-(L/\beta)\,(1-\alpha)\,\alpha^{\frac{\alpha}{1-\alpha}}$, indepen-

[7] See Cysne and Turchick (2012), propositions 1 and 2.

dent from δ. Therefore

$$
\begin{aligned}
\frac{\partial \kappa_1}{\partial \delta} &= -\frac{\partial \kappa_2}{\partial \delta} = \frac{\alpha\delta - (1-\alpha+\delta)\,\alpha}{(\alpha\delta)^2}(r+m) + \frac{1-\alpha+\delta}{\alpha\delta}\frac{\partial r}{\partial \delta} \\
&= \frac{-(1-\alpha)^2(1+\delta) + (1-\alpha+\delta)(1-\alpha(1+\delta))}{\alpha\delta^2}\frac{r+m}{(1-\alpha)(1+\delta)} \\
&= \frac{-(1-\alpha)(1+\delta) + 1-\alpha+\delta(1-\alpha(1+\delta))}{\alpha\delta^2}\frac{r+m}{(1-\alpha)(1+\delta)} \\
&= \frac{\delta(-(1-\alpha)+1-\alpha(1+\delta))}{\alpha\delta^2}\frac{r+m}{(1-\alpha)(1+\delta)} = -\frac{r+m}{(1-\alpha)(1+\delta)}.
\end{aligned}
$$

Secondly, implicit differentiation of $\kappa_1\zeta^2 + (\kappa_2 - m)\,\zeta + m = 0$ gives

$$
\begin{aligned}
\frac{\partial \zeta}{\partial \delta} &= -\frac{\frac{\partial \kappa_1}{\partial \delta}\zeta^2 + \frac{\partial \kappa_2}{\partial \delta}\zeta}{2\kappa_1\zeta + \kappa_2 - m} = -\frac{\frac{\partial \kappa_1}{\partial \delta}\zeta^2 - \frac{\partial \kappa_1}{\partial \delta}\zeta}{2\kappa_1\zeta + \kappa_2 - m} \\
&= \frac{\zeta^2(1-\zeta)}{2\kappa_1\zeta^2 + (\kappa_2 - m)\,\zeta}\frac{\partial \kappa_1}{\partial \delta} = -\frac{\zeta^2(1-\zeta)}{\kappa_1\zeta^2 - m}\frac{r+m}{(1-\alpha)(1+\delta)}.
\end{aligned}
$$

It is also instrumental (see (14)) to compute

$$
\begin{aligned}
\frac{\partial(\kappa_1\zeta - m - \gamma)}{\partial \delta} &= \frac{\partial \kappa_1}{\partial \delta}\zeta + \kappa_1\frac{\partial \zeta}{\partial \delta} - \frac{\partial \gamma}{\partial \delta} \\
&= \left(-\zeta - \kappa_1\frac{\zeta^2(1-\zeta)}{\kappa_1\zeta^2 - m} - \frac{1-\alpha(1+\delta)}{\delta}\right)\frac{r+m}{(1-\alpha)(1+\delta)} \\
&= -\left(\frac{\zeta\left(\kappa_1\zeta^2 - m\right) + \kappa_1\zeta^2(1-\zeta)}{\kappa_1\zeta^2 - m} + \frac{1-\alpha(1+\delta)}{\delta}\right)\frac{r+m}{(1-\alpha)(1+\delta)} \\
&= -\left(\frac{\zeta(\kappa_1\zeta - m)}{\kappa_1\zeta^2 - m} + \frac{1-\alpha(1+\delta)}{\delta}\right)\frac{r+m}{(1-\alpha)(1+\delta)}
\end{aligned}
$$

and

$$
\frac{\partial\left(1-\frac{g_0}{\zeta}\right)}{\partial \delta} = \frac{g_0}{\zeta^2}\frac{\partial \zeta}{\partial \delta} = -\frac{1-\zeta}{\kappa_1\zeta^2 - m}g_0\frac{r+m}{(1-\alpha)(1+\delta)}.
$$

Thus

$$
\begin{aligned}
\frac{\partial h_0}{\partial \delta} &= \frac{\partial\left(\kappa_1\zeta - m - \gamma\right)}{\partial \delta}\left(1-\frac{g_0}{\zeta}\right) + \left(\kappa_1\zeta - m - \gamma\right)\frac{\partial\left(1-\frac{g_0}{\zeta}\right)}{\partial \delta} \\
&= -\left[\begin{array}{c} \left(\frac{\zeta(\kappa_1\zeta-m)}{\kappa_1\zeta^2-m}+\frac{1-\alpha(1+\delta)}{\delta}\right)\left(1-\frac{g_0}{\zeta}\right)+ \\ \left(\kappa_1\zeta-m-\gamma\right)\frac{1-\zeta}{\kappa_1\zeta^2-m}g_0 \end{array}\right]\frac{r+m}{(1-\alpha)(1+\delta)} \\
&= -\left[\begin{array}{c} \frac{\zeta(\kappa_1\zeta-m)}{\kappa_1\zeta^2-m}+\frac{1-\alpha(1+\delta)}{\delta}- \\ \left(\frac{\kappa_1\zeta-m}{\kappa_1\zeta^2-m}+\frac{1}{\zeta}\frac{1-\alpha(1+\delta)}{\delta}-\frac{(1-\zeta)(\kappa_1\zeta-m-\gamma)}{\kappa_1\zeta^2-m}\right)g_0 \end{array}\right]\frac{r+m}{(1-\alpha)(1+\delta)} \\
&= -\left[\begin{array}{c} \frac{\zeta(\kappa_1\zeta-m)}{\kappa_1\zeta^2-m}+\frac{1-\alpha(1+\delta)}{\delta}- \\ \left(\frac{\zeta(\kappa_1\zeta-m)+(1-\zeta)\gamma}{\kappa_1\zeta^2-m}+\frac{1}{\zeta}\frac{1-\alpha(1+\delta)}{\delta}\right)g_0 \end{array}\right]\frac{r+m}{(1-\alpha)(1+\delta)}. \qquad (16)
\end{aligned}
$$

The last term in square brackets is an affine function of g_0 taking, at the extreme $g_0 = 0$, the value

$$
\frac{\zeta\left(\kappa_1\zeta - m\right)}{\kappa_1\zeta^2 - m} + \frac{1-\alpha\left(1+\delta\right)}{\delta},
$$

which is positive (we know that $\kappa_1\zeta - m \geq \gamma$ from (14), and $\gamma \geq 0$ within F). At the extreme $g_0 = 1$, it equals

$$
-\frac{\gamma\left(1-\zeta\right)}{\kappa_1\zeta^2 - m} + \left(1-\frac{1}{\zeta}\right)\frac{1-\alpha\left(1+\delta\right)}{\delta},
$$

again positive. Hence $\partial h_0/\partial\delta < 0$. ■

The policymaker thus faces two simultaneous IPR protection tradeoffs in this model: one in terms of patent length, and another in terms of patent breadth. The next section solves this bidimensional problem.

4 Optimal patent length and breadth

The inverse-length/breadth pair (m, δ) represents the policy vector of the model, and F the set where it can be chosen. The parameters L, α, and β are exogenous to the government.

Although m is a primitive of the model, one could obviously choose to work with T instead.[8] We choose m simply for ease of calculation, but we express our results, especially the numerical ones, in terms of T rather than m, for its more direct association with an observable variable.

Now consider that the government can make a once-and-for-all change in its IPR protection policy, from $(m, \delta) \in F$ to $(m', \delta') \in F$ (so that the evolution of A_c is now governed by (3) with parameter m').[9] For simplicity, say this change takes place at time 0.

Imagine the economy starts in its steady state $\left(\overline{g}, \overline{h}\right)$. Since g cannot jump, $g_0' = \overline{g}$. The interest rate must jump immediately to its new long-term level according to the new arbitrage relation

$$r' + m' = \frac{L}{\beta}\delta'\left(\frac{\alpha}{1+\delta'}\right)^{\frac{1}{1-\alpha}}. \tag{17}$$

The dynamical system coefficients κ_1 and κ_2 become $\kappa_1' = \left(1 - \alpha + \delta' - (1-\alpha)(1+\delta')^{\frac{1}{1-\alpha}}\right)(r' + m') / (\alpha\delta')$ and $\kappa_2' = -(1 - \alpha + \delta')(r' + m') / (\alpha\delta')$.[10] The rate of growth of consumption γ is given by the Euler equation

$$\gamma' = r' - \rho, \tag{18}$$

and the new steady state is $\left(\overline{g}', \overline{h}'\right) = (m' / (m' + \gamma'), -\kappa_1' m' / (m' + \gamma') - \gamma' - \kappa_2')$. At the moment the government employs this policy change, h jumps from $\overline{h}$ to

$$h_0' = \frac{g_0' - \zeta'}{\overline{g}' - \zeta'}\overline{h}', \tag{19}$$

[8] This is a simple change of variables: given (m, δ), (8) and (7) give T, whereas given (T, δ), m can be derived from those same two equations.

[9] An alternate scenario, in which the government cannot change (m, δ) directly, will incidentally be dealt with in section 6 ahead.

[10] Note that, differently from Cysne and Turchick (2012), where δ is set constant at its maximal level, here the change in m is not perfectly mirrored by the change in r, that is, $r' + m' \neq r + m$ (unless $\delta' = \delta$).

where $\zeta' = \left(m' - \kappa_2' + \sqrt{\left(m' - \kappa_2'\right)^2 - 4\kappa_1' m'}\right) / \left(2\kappa_1'\right)$. As before, we can also write

$$h_0' = \left(\kappa_1' \zeta' - m' - \gamma'\right) \left(1 - \frac{g_0'}{\zeta'}\right). \tag{20}$$

Our main goal is to find the optimal government policy vector $(m^*, \delta^*) = (m', \delta')$. For this purpose, we substitute $\beta A(0) h_0' e^{\gamma' t}/L$ for c in the representative agent's utility function, which yields the total intertemporal utility level associated with (m', δ') accumulated throughout the new equilibrium path: $U^*(m', \delta') = (\log(\beta A(0)/L) + \log h_0' + \gamma'/\rho)/\rho$. Thus the government must solve the problem

$$\max_{(m',\delta') \in F} H(m', \delta', g_0'),$$

where $H : F \times [0, 1] \to \mathbb{R}$ is given by

$$H(m', \delta', g_0') := \log h_0' + \frac{\gamma'}{\rho} \tag{21}$$

(h_0' and γ' are related to (m', δ', g_0') through the expressions above). Since H is continuous and F is compact, existence of (m^*, δ^*) is guaranteed by the Weierstrass Theorem.

4.1 Optimal length

It has been proven (Cysne and Turchick, 2012, proposition 3) that, in the case of a lab-equipment model with logarithmic utility function, the optimal patent length T^* is necessarily infinite, irrespective of the initial length T. This result should also be expected in the present more general framework if it happened to be the case that the government was initially offering maximal IPR protection in terms of patent breadth ($\delta = 1/\alpha - 1$), as in that paper. But if $\delta < 1/\alpha - 1$, it is unclear what the best course of action for the government might be. Our framework allows us to consider various policy mix possibilities, such as weakening/strengthening IPR protection in terms of length while strengthening/weakening it in terms of breadth.

The following proposition partially answers this question.

Proposition 2 *In this economy, the optimal patent length T^* is infinite, regardless of the current patent length T or the current patent breadth level δ.*

Proof. Fix any $\delta' \in (\underline{\delta}, 1/\alpha - 1]$ (and $g_0' = \overline{g}$ is given). From (8), we must show that $m^* = 0$ (it is immediate to see that $(0, \delta') \in F$). For this purpose, it is sufficient to verify that $D_1 H < 0$. We do so in three steps. Let $m' \in \mathbb{R}_+$ be such that $(m', \delta') \in F$.

Step (i): $D_1 H(m', \delta', g_0') \leq D_1 H(m', \delta', 0)$.

Expression (21), the law of H, together with (18) and (17), yields $D_1 H(m', \delta', g_0') = (1/h_0')\, \partial h_0' / \partial m' - 1/\rho$. The term $-m' - \gamma'$ in expression 20 for h_0' has actually no dependence on m', again from (18) and (17). The same is true for κ_1' and κ_2'. Thus we have

$$\frac{\partial h_0'}{\partial m'} = \kappa_1' \frac{\partial \zeta'}{\partial m'} \left(1 - \frac{g_0'}{\zeta'}\right) + (\kappa_1' \zeta' - m' - \gamma') \frac{g_0'}{\zeta'^2} \frac{\partial \zeta'}{\partial m'} = \left(\kappa_1' - \frac{m' + \gamma'}{\zeta'^2} g_0'\right) \frac{\partial \zeta'}{\partial m'}$$

and, from (19),

$$D_1 H(m', \delta', g_0') = \frac{1}{\kappa_1' \zeta' - m' - \gamma'} \frac{\kappa_1' - \frac{m' + \gamma'}{\zeta'^2} g_0'}{1 - \frac{1}{\zeta'} g_0'} \frac{\partial \zeta'}{\partial m'} - \frac{1}{\rho}.$$

The fraction above with g_0' in both the numerator and denominator can be seen to be increasing in g_0'.[11] The derivative $\partial \zeta' / \partial m'$, in its turn, comes from implicit differentiation of

[11]It is straightforward to note that $(a + bx) / (c + dx)$ is increasing in x if and only if $bc - ad \geq 0$. For the case at stake, it must be the case that $\kappa_1' \zeta' - m' - \gamma' \geq 0$, a fact that follows immediately from (20) ($(m', \delta') \in F$ implies $h_0' \geq 0$, as seen in the previous section).

$\kappa_1'\zeta'^2 + (\kappa_2' - m')\,\zeta' + m' = 0$:

$$\frac{\partial \zeta'}{\partial m'} = -\frac{1-\zeta'}{\kappa_2' - m' + 2\kappa_1'\zeta'} = -\frac{\zeta'\,(1-\zeta')}{\kappa_1'\zeta'^2 - m'} < 0.$$

Therefore $D_1 H\,(m', \delta', g_0') \leq D_1 H\,(m', \delta', 0)$.

Step (ii): $D_1 H\,(m', \delta', 0) \leq D_1 H\,(0, \delta', 0)$.

Putting $g_0' = 0$ in the expression for $D_1 H\,(m', \delta', g_0')$ gives

$$D_1 H\,(m', \delta', 0) = \frac{-\kappa_1'}{\kappa_1'\zeta' - m' - \gamma'}\left(-\frac{\partial \zeta'}{\partial m'}\right) - \frac{1}{\rho}.$$

As explained in step (i), the first fraction (which is positive) depends on m' through ζ' only, whence it decreases with m'. The term in parentheses is also positive, and in order to see that it also decreases with m', one must calculate

$$\begin{aligned}
\frac{\partial^2 \zeta'}{\partial m'^2} &= -\frac{\left(\frac{\partial \zeta'}{\partial m'} - 2\zeta'\frac{\partial \zeta'}{\partial m'}\right)\left(\kappa_1'\zeta'^2 - m'\right) - \zeta'\,(1-\zeta')\left(2\kappa_1'\zeta'\frac{\partial \zeta'}{\partial m'} - 1\right)}{\left(\kappa_1'\zeta'^2 - m'\right)^2} \\
&= -\frac{\left[-\kappa_1'\zeta'^2 - m'\,(1-2\zeta')\right]\frac{\partial \zeta'}{\partial m'} + \zeta'\,(1-\zeta')}{\left(\kappa_1'\zeta'^2 - m'\right)^2} = -\zeta'\frac{(\kappa_2' + m')\,\frac{\partial \zeta'}{\partial m'} + 1 - \zeta'}{\left(\kappa_1'\zeta'^2 - m'\right)^2}
\end{aligned}$$

and notice that this is positive, since

$$\begin{aligned}
\kappa_2' + m' &= -\frac{1-\alpha+\delta'}{\alpha\delta'}\,(r' + m') + m' \leq -\frac{1-\alpha+\delta'}{\alpha\delta'}\,(r' + m') + r' + m' \\
&= -\frac{(1-\alpha)\,(1+\delta')}{\alpha\delta'}\,(r' + m') < 0.
\end{aligned}$$

Thus $D_1 H\,(m', \delta', 0)$ is decreasing in m, and $D_1 H\,(m', \delta', 0) \leq D_1 H\,(0, \delta', 0)$.

Step (iii): $D_1 H\,(0, \delta', 0) < 0$.

For $m' = 0$, we get $\zeta' = \left(-\kappa_2' + \sqrt{\kappa_2'^2}\right)/\,(2\kappa_1') = -\kappa_2'/\kappa_1'$ and $\overline{h}' = -\gamma' - \kappa_2'$ (positive within F as seen in section 2), so

$$\begin{aligned} D_1 H(0, \delta', 0) &= \frac{-\kappa_1'}{\kappa_1' \zeta' - \gamma'} \frac{\zeta' (1 - \zeta')}{\kappa_1' \zeta'^2} - \frac{1}{\rho} = \frac{1}{-\gamma' - \kappa_2'} \left(\frac{\kappa_1' + \kappa_2'}{\kappa_2'} + \frac{\gamma' + \kappa_2'}{\rho} \right) \\ &= \frac{1}{\overline{h}'} \left(\frac{\kappa_1' + \kappa_2'}{\kappa_2'} + \frac{r' - \rho - \frac{1-\alpha+\delta'}{\alpha\delta'} r'}{\rho} \right) \\ &\leq \frac{1}{\overline{h}'} \left[\frac{(1-\alpha)(1+\delta')^{\frac{1}{1-\alpha}}}{1-\alpha+\delta'} - \frac{(1-\alpha)(1+\delta')}{\alpha\delta'} - 1 \right], \end{aligned}$$

where (18) and the fact that $r' \geq \rho$ (or $\gamma' \geq 0$) were employed in the second and third lines, respectively. The second fraction in the square brackets must be greater or equal to 1, otherwise $1 + \delta' < \alpha\delta' / (1 - \alpha) \leq \alpha (1/\alpha - 1) / (1 - \alpha) = 1$, a contradiction. The first fraction grows with δ', and at its highest (at $\delta' = 1/\alpha - 1$) it equals $\alpha^{-\frac{\alpha}{1-\alpha}} / (1 + \alpha)$. This can be seen to be strictly less than 2. In fact, one would otherwise have

$$\begin{aligned} 2(1+\alpha) &\leq \alpha^{-\frac{\alpha}{1-\alpha}} \therefore (2(1+\alpha))^{1-\alpha} \leq \alpha^{-\alpha} \therefore \\ 2(1+\alpha) &\leq \left(\frac{2(1+\alpha)}{\alpha} \right)^{\alpha} \therefore (2(1+\alpha))^{\frac{1}{\alpha}} \leq 2\left(1 + \frac{1}{\alpha}\right), \end{aligned}$$

which contradicts Bernoulli's inequality, according to which $2(1 + 1/\alpha) < 2(1+1)^{\frac{1}{\alpha}} = 2^{1+\frac{1}{\alpha}} = 2^{\frac{1}{\alpha}} (1 + (1/\alpha)\alpha) < (2(1+\alpha))^{\frac{1}{\alpha}}$. Thus $D_1 H(0, \delta', 0) < (2 - 1 - 1) / \overline{h}' = 0$.

Finally putting all the pieces together, for $(m', \delta') \in F$, $D_1 H(m', \delta', g_0') \leq D_1 H(m', \delta', 0) \leq D_1 H(0, \delta', 0) < 0$. Hence we obtain the (unique) corner solution $m^* = 0$. ■

Proposition 2 allows for an automatic reduction in the dimensionality of the policymaker's problem. No longer does he have to look jointly for the socially optimal policy vector (m^*, δ^*), and he can simply plug in $m^* = 0$ into all the expressions in order to find δ^*. This simplification is a feature of horizontal innovation models, to some extent linked to the fact that capital goods never become obsolete in this framework.[12]

[12] However, this connection is not the full story. If, instead of being fixed at 1, the elasticity of intertemporal

4.2 Optimal breadth

We now focus our attention on the government's second policy parameter, patent breadth. We first check that it is always the case that the maximal level $1/\alpha - 1$ (that royalty fee preferred by innovators when government is unable to fix breadth) is suboptimal. Thus the government will actually not choose in accordance with the model in Cysne and Turchick (2012), where patent breadth is fixed at a maximum, and will specify a breadth level $\delta' < 1/\alpha - 1$ instead. In this way, it will be able to trade off a modest negative long-term effect on growth (the term involving $\partial\gamma'/\partial\delta'$ in expression (22) ahead) for a substantial positive short-term effect on consumption (the term involving $\partial h_0'/\partial\delta'$).

Proposition 3 *In this economy, the social optimal patent breadth level δ^* is strictly lower than the maximal breadth level $1/\alpha - 1$, regardless of the current patent length T or the current patent breadth level δ.*

Proof. We already know, from Proposition 2, that $m^* = 0$. In order to prove that $\delta^* < 1/\alpha - 1$, it therefore suffices to show that $D_2H(0, 1/\alpha - 1, g_0') < 0$, that is,

$$\left[\frac{1}{h_0'}\frac{\partial h_0'}{\partial \delta'} + \frac{1}{\rho}\frac{\partial \gamma'}{\partial \delta'}\right|_{\delta'=\frac{1}{\alpha}-1} < 0 \tag{22}$$

(from the definition of $\underline{\delta}$, it is clear that $(0, \delta') \in F, \forall \delta' \in (\underline{\delta}, 1/\alpha - 1]$, so it is adequate to think of this derivative as a left derivative). Here, as in the proof of Proposition 1 (see (15) and (16)), and using $\delta' = 1/\alpha - 1$ and $m' = m^* = 0$, we get

$$\left.\frac{\partial \gamma'}{\partial \delta'}\right|_{\delta'=\frac{1}{\alpha}-1} = \left.\frac{\partial r'}{\partial \delta'}\right|_{\delta'=\frac{1}{\alpha}-1} = \frac{1-\alpha\,(1+\delta')}{\delta'}\,\frac{r'}{(1-\alpha)\,(1+\delta')} = 0$$

substitution were allowed to take on especially low values (about 1/5 or less, see the figures in Cysne and Turchick, 2012), then consumers would be less keen to sacrifice current for future consumption, meaning T^* could also be finite.

and

$$\left.\frac{\partial h_0'}{\partial \delta'}\right|_{\delta'=\frac{1}{\alpha}-1} = -\left[\begin{array}{c}\frac{\kappa_1'\zeta'^2}{\kappa_1\zeta^2}+\frac{1-\alpha(1+\delta')}{\delta'}- \\ \left(\frac{\kappa_1'\zeta'^2+(1-\zeta')\gamma'}{\kappa_1'\zeta'^2}+\frac{1}{\zeta'}\frac{1-\alpha(1+\delta')}{\delta'}\right)g_0'\end{array}\right]\frac{r'}{(1-\alpha)(1+\delta')}$$

$$= -\left[1-\frac{\kappa_1'\zeta'^2+(1-\zeta')\gamma'}{\kappa_1'\zeta'^2}g_0'\right]\frac{r'}{(1-\alpha)(1+\delta')}.$$

From (17) and Proposition 2 we know that r' must be (strictly) positive. The term inside square brackets is an affine function of g_0', which at the extreme $g_0'=0$ takes on the value 1, and at $g_0'=1$ equals $-(1-\zeta')\gamma'/\left(\kappa_1'\zeta'^2\right)$, which is also positive since $\gamma'=m'+\gamma'>0$ for $(m',\delta')\in F$ (indeed, (17) and our assumption that $\rho<(L/\beta)(1/\alpha-1)\alpha^{\frac{2}{1-\alpha}}$ give $\gamma'=r'-\rho=r'+m'-\rho=(L/\beta)(1/\alpha-1)\alpha^{\frac{2}{1-\alpha}}-\rho>0$). So $\partial h_0'/\partial\delta'|_{\delta'=1/\alpha-1}$ is negative, and so is $D_2H(0,1/\alpha-1,g_0')$, thus ending the proof. ■

Having established its interiority (relative to the $[0,1/\alpha-1]$ interval), the optimal level of patent breadth can be found by plugging (15), (16) and $m^*=0$ into the FOC

$$\left[\frac{1}{h_0'}\frac{\partial h_0'}{\partial\delta'}+\frac{1}{\rho}\frac{\partial\gamma'}{\partial\delta'}\right|_{\delta'=\delta^*}=0$$

(or, if this equality does not hold for any $\delta^*\in(\underline{\delta},1/\alpha-1)$, then $\delta^*=\underline{\delta}$).

Factoring out the term $r'/((1-\alpha)(1+\delta^*))$ of both $\partial h_0'/\partial\delta'$ and $\partial\gamma'/\partial\delta'$, and making use of (20) and the fact that $\zeta'=-\kappa_2'/\kappa_1'$ when $m'=0$, leads to the implicit formula

$$\frac{1}{(\kappa_2'+\gamma')\left(1+\frac{\kappa_1'}{\kappa_2'}g_0'\right)}\left[1+\frac{1-\alpha(1+\delta^*)}{\delta^*}-\left(1+\frac{\kappa_1'+\kappa_2'}{\kappa_2'^2}\gamma'-\frac{\kappa_1'}{\kappa_2'}\frac{1-\alpha(1+\delta^*)}{\delta^*}\right)g_0'\right]+$$
$$\frac{1}{\rho}\frac{1-\alpha(1+\delta^*)}{\delta^*}=0. \tag{23}$$

It should be noted that δ^* appears also inside the terms κ_1', κ_2', and $\gamma'=r'-\rho=(L/\beta)\delta^*(\alpha/(1+\delta^*))^{\frac{1}{1-\alpha}}-\rho$ above.

We exemplify these results with the baseline case parameter values used in Kwan and Lai (2003), $\alpha = 0.625$, $r = 0.065/\text{year}$, $\gamma = 0.016/\text{year}$, and $\delta = 1/\alpha - 1 = 0.600$. We also use $T = 20$ years, the current patent term in the United States (that work used the previous length of 17 years).[13] The socially optimal patent breadth is calculated as approximately 0.493. Table 1 compares a situation in which government does not alter the policy vector (T, δ) (first column) with the situation in which it can alter it only in the first variable (second column), and then with the situation in which it is free to alter both variables (as is shown in Propositions 2 and 3, it will do so by choosing a maximal patent length but a nonmaximal patent breadth).

Table 1

$\alpha = 0.625$, $r = 0.065$, $\gamma = 0.016$, $T = 20$ $(m \approx 0.024)$, $\delta = 0.600$

New length/breadth (T', δ')	$(20, 0.600)$	$(\infty, 0.600)$	$(\infty, 0.493)$
Growth rate γ':	0.0160	0.0404	0.0393
Short-run "consumption" h_0':	0.2650	0.2322	0.2421
Long-run "consumption" $\overline{h}'$:	0.2650	0.1920	0.2095
"Utility" $H(m(T', r'), \delta', \overline{g})$:	-1.0017	-0.6369	-0.6166

As explained in Proposition 1, the obtaining of a higher welfare through revision of the patent breadth level (-0.6166 versus -0.6369) is accompanied by a larger steady-state (scaled-)consumption level (0.2095 versus 0.1920) and a bit of a sacrifice in the growth rate (0.11 percentage points). This higher welfare happens due to h_0' not needing to be cut down as severely as in the case corresponding to a fixed maximal level of patent breadth. The utility gain from -1.0017 to -0.6166 means that the drop in steady-state normalized

[13] For future reference, we note that the values of ρ and L/β consistent with this equilibrium are 0.049 and 1.826, respectively (to see this, one may use (1), and substitute (4) in (7)).

consumption $h = C/(\beta A)$ from 0.2650 to 0.2095 is more than compensated by the larger growth rate of the economy (3.93% versus 1.60%).

The two trajectories of h (corresponding to the two last columns of the previous table) can be compared in Figure 1, which also plots the trajectory corresponding to government inaction.

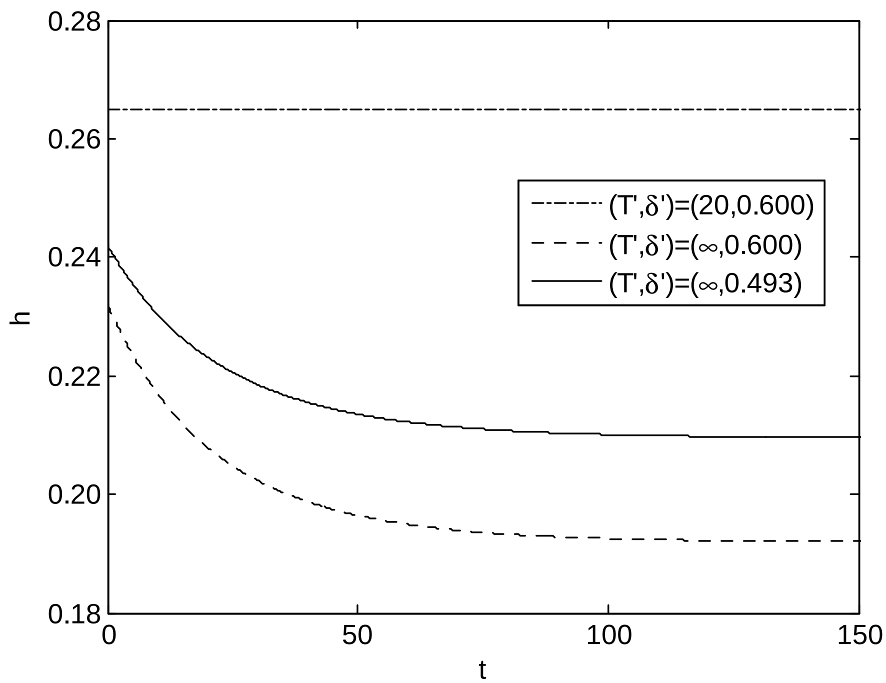

Figure 1. Time paths for h.

Figure 2 plots $c/A(0) = \beta h_0' e^{\gamma' t}/L$, thus allowing for a better visualization of the beneficial effect of being able to cut down patent breadth. Although we know the dashed curve has a slightly superior growth rate than the solid curve (they shall meet around year 40), the latter involves a lesser sacrifice in terms of immediate consumption than the former, hence being able to generate more welfare. It might be observed that, while the $(\infty, 0.600)$ policy takes about five and a half years to pay off in terms of consumption only, the adoption of

$(\infty, 0.493)$, which corresponds to roughly the same growth rate but also to a significantly reduced drop in immediate consumption, pays off earlier, in less than four years.

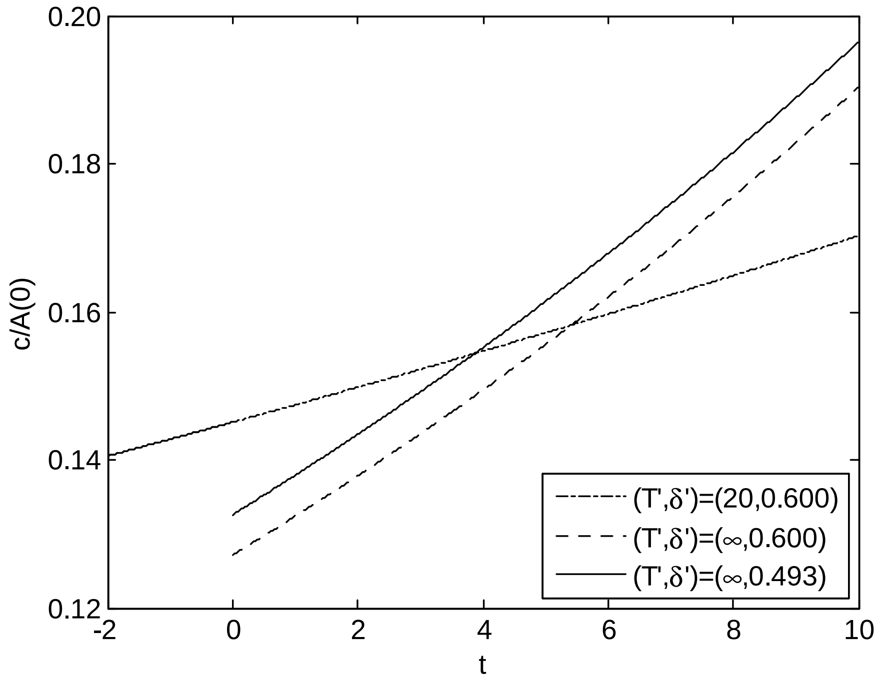

Figure 2. Time paths of consumption per capita (in terms of $A(0)$).

Figure 3 shows the behavior of the optimal patent breadth level with respect to both initial patent length and breadth. As we can see, δ^*depends on δ and T. The higher these initial IPR protection levels, the lower needs the optimal patent breadth be. This result is in line with findings regarding optimal patent length policy in Kwan and Lai (2003) and Cysne and Turchick (2012).

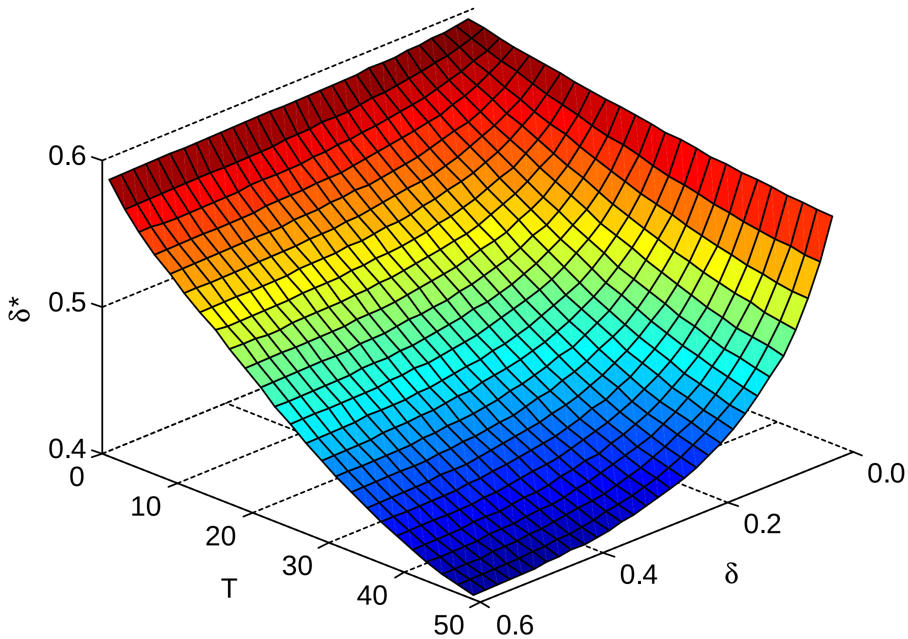

Figure 3. Optimal patent breadth level.

5 Time-consistent policies

A delicate issue in the present model is that of intertemporal consistency. Once the government chooses a high enough level of IPR protection in order to ensure dynamic efficiency gains, it is tempted to overlook product imitation and try to obtain static efficiency gains as well. For such a government with credibility or reputation issues, the only surviving steady-state equilibrium would be one with $(m, \delta) = (m^*, \delta^*)$.

From Proposition 2, it is clear that the optimal patent length policy is necessarily time consistent (with $m = m^* = 0$). Thus, the no-commitment-equilibrium policy mix $(m, \delta) = (0, \delta^{**})$ must be a pair that solves (23). Using $g_0' = \bar{g} = m/(m+\gamma) = 0$, that expression becomes

$$\frac{1}{\kappa_2+\gamma}\left[1+\frac{1-\alpha\left(1+\delta^{**}\right)}{\delta^{**}}\right]+\frac{1}{\rho}\frac{1-\alpha\left(1+\delta^{**}\right)}{\delta^{**}}=0,$$

where

$$\kappa_2+\gamma=-\frac{1-\alpha+\delta^{**}}{\alpha\delta^{**}}r+r-\rho=\left(1-\frac{1-\alpha+\delta^{**}}{\alpha\delta^{**}}\right)r-\rho=-\frac{\left(1-\alpha\right)\left(1+\delta^{**}\right)}{\alpha\delta^{**}}r-\rho$$

and

$$r=\frac{L}{\beta}\delta^{**}\left(\frac{\alpha}{1+\delta^{**}}\right)^{\frac{1}{1-\alpha}}.$$

With respect to the aforementioned parameter values, we get $\delta^{**}\approx 0.415$. That is, the optimal value for license fees $\delta^{*}\approx 0.493$ found in the previous section must be cut in about 15%, if the government has no reputation in keeping its promises. Table 2 shows the welfare cost for society of government not having a commitment mechanism, or having such a bad reputation that the announced policy has no credibility. The first column corresponds to no policy change at all, the second to optimal change when commitment is not an issue (the same as in Table 1), and the third to a change to the no-commitment-equilibrium policy.

Table 2

$\alpha=0.625$, $r=0.065$, $\gamma=0.016$, $T=20$ ($m\approx 0.024$), $\delta=0.600$

New length/breadth (T',δ')	$(20, 0.600)$	$(\infty, 0.493)$	$(\infty, 0.415)$
Growth rate γ':	0.0160	0.0393	0.0368
Short-run "consumption" h_0':	0.2650	0.2421	0.2513
Long-run "consumption" $\overline{h}'$:	0.2650	0.2095	0.2245
"Utility" $H\left(m\left(T',r'\right),\delta',\overline{g}\right)$:	-1.0017	-0.6166	-0.6311

As expected, a government with credibility issues will choose a lower level of patent breadth, when compared to the equilibrium with commitment of the previous section. In this case, its inability to commit may be mistaken for myopic preferences.

6 Social planner's problem

In this section, we assess the issue of social optimality of the equilibrium policy described in section 4 for the decentralized economy. The analysis borrows from and extends the one worked out in Barro and Sala-i-Martin (2004, chapter 6).

In order to do that, we compare it to the centrally planned economy, where a social planner maximizes the utility of households, $U = \int_0^\infty e^{-\rho t} \log(C_t/L)\, dt$, with respect to the resource contraint of the economy, (9).

An application of Pontryagin's Maximum Principle yields

$$X_s = AL\alpha^{\frac{1}{1-\alpha}}$$

(by differentiating the Hamiltonian of the problem with respect to X) and

$$\gamma_s = r_s - \rho$$

(by differentiating the Hamiltonian with respect to C), where

$$r_s := \frac{L}{\beta}\left(\frac{1}{\alpha} - 1\right)\alpha^{\frac{1}{1-\alpha}}$$

(the additive inverse of the growth rate of the costate variable).

A simple comparison of the production of intermediate goods here and in the decentralized economy gives a measure of the static inefficiencies stemming from inventors' market power: $X = A_c x_c + (A - A_c)\, x_m < A_c x_c + (A - A_c)\, x_c = A x_c = AL\alpha^{\frac{1}{1-\alpha}} = X_s$ (using

$x_m < x_c$, since $\delta > 0$). A comparison of growth rates evinces the dynamic inefficiency:

$$\begin{aligned}\gamma &= r-\rho=\frac{L}{\beta}\delta\left(\frac{\alpha}{1+\delta}\right)^{\frac{1}{1-\alpha}}-m-\rho\leq\frac{L}{\beta}\left(\frac{1}{\alpha}-1\right)\left(\frac{\alpha}{1+\delta}\right)^{\frac{1}{1-\alpha}}-\rho\\ &< \frac{L}{\beta}\left(\frac{1}{\alpha}-1\right)\alpha^{\frac{1}{1-\alpha}}-\rho=r_s-\rho=\gamma_s\end{aligned}$$

(since $\delta > 0$). It is thus clear that the decentralized equilibrium is not efficient. No licensing at all brings about dynamic inefficiencies (a lack of incentives for inventors, who are subject to the technology imitation equation (3)), while positive licensing implies static deadweight losses.

Nevertheless, it is possible to obtain an efficient outcome in the decentralized economy, as long as the government can tax consumers in a lumpsum fashion and use the proceeds to subsidize both the R&D sector and the purchase of intermediate goods by the final goods sector. In fact, given any (m, δ), even in a scenario in which the policymaker is unable to directly mandate on the level of patent breadth δ, as explained in the introduction, he/she may choose $m' = 0$ (the reasoning follows the lines of Proposition 2), then (i) subsidize by the amount δ the purchase of each unit of yet monopolized capital goods, so that the $1+\delta$ entering the expression of x_m in (5) becomes $1+\delta-\delta = 1$, and $x_m = x_c = L\alpha^{\frac{1}{1-\alpha}}$, so that $X' = A_c x_c + (A - A_c)\, x_m = AL\alpha^{\frac{1}{1-\alpha}} = X_s$, and (ii) subsidize R&D by means of a lower invention cost β' such that $\gamma' = \gamma_s$, i.e., we must have

$$\beta'=\frac{\pi'}{r'+m'}=\frac{\delta x_m}{r'}=\frac{\delta x_c}{\gamma'+\rho}=\frac{\delta L\alpha^{\frac{1}{1-\alpha}}}{\gamma_s+\rho}=\frac{\delta L\alpha^{\frac{1}{1-\alpha}}}{r_s}=\frac{\delta L\alpha^{\frac{1}{1-\alpha}}}{\frac{L}{\beta}\left(\frac{1}{\alpha}-1\right)\alpha^{\frac{1}{1-\alpha}}}=\beta\frac{\delta}{\frac{1}{\alpha}-1},$$

which will be lower than β (unless $\delta = 1/\alpha - 1$, in which case they are equal). That is, in order to cope with two sources of inefficiencies, the policymaker must resort to two policy instruments.

7 Conclusion

Although the issue of intellectual property rights protection, both for patent lengths and breadths, provides policymakers with a short-run-vs.-long-run tradeoff (Cysne and Turchick, 2012, and Proposition 1 above), we have shown that their solutions are qualitatively different (a corner one and an interior one, respectively). As confirmed by our numerical analysis, by fixing a lower patent breadth level, the government is able to trade off a modest negative long-term effect on growth for a substantial positive short-term effect on consumption, whence it cannot simply discard the short-run from the analysis. In order to rigorously prove this, we have relied on the horizontal innovation lab-equipment model of R&D-based growth coupled with an exogenous imitation rate (so that patent lengths are of a probabilistic nature). Such a proof was made possible (Propositions 2 and 3) once a closed-form solution to the model's dynamics was derived.

On the one hand, the analysis in this work confirms and strengthens a conclusion obtained in Cysne and Turchick (2012), in the sense that, even when there is plenty of room to increase IPR protection in terms of breadth, infinite patent lengths will still be optimal. On the other hand, it distances itself from the conclusion in that paper in that it finds that the IPR protection tradeoff in terms of breadth will not yield a corner solution of favoring only innovators (there, δ is fixed at the maximum level of $1/\alpha - 1$, which, as we have shown, is suboptimal).

Due to static and dynamic inefficiencies, the decentralized equilibrium is not socially optimal. However, these inefficiencies are reconcilable if the policymaker subsidizes R&D and the purchase of patent-protected capital goods. An additional source of inefficiency is the lack of a commitment mechanism for the government, and we have shown how this inefficiency can be gauged.

The proposed framework for the analysis of the optimal mix of IPR protection policies is admittedly a very simple one, and obviously many extensions are possible, such as considering different notions of patent breadth, allowing for a lower-than-unity elasticity of intertemporal substitution (thus challenging the mentioned optimality of infinite patent lifetimes), and including some actual modelling of government reputation into the analysis. But the present simplicity is in itself a feature that has allowed us to gain some insight on this problem, formally test some of our intuitions, and to have a benchmark for the assessment of results from any extension of this model.

References

Barro, R.J., Sala-i-Martin, X., 2004. Economic Growth, 2nd ed. MIT Press, Cambridge, MA.

Becker, S.L., Lu, J., 2009. Royalty rate and industry structure: Some cross-industry evidence. Available at: http://dx.doi.org/10.2139/ssrn.1447997.

Bessen, J., Maskin, E., 2009. Sequential innovation, patents, and imitation. RAND Journal of Economics 40, 611-635.

Chen, M.X., Iyigun, M., 2011. Patent protection and strategic delays in technology development: Implications for economic growth. Southern Economic Journal 78, 211-232.

Chu, A.-C., Furukawa, Y., 2011. On the optimal mix of patent instruments. Journal of Economic Dynamics & Control 35, 1964-1975.

Cysne, R.P., Turchick, D., 2012. Intellectual property rights protection and endogenous economic growth revisited. Journal of Economic Dynamics & Control 36, 851-861.

Erkal, N., 2005. The decision to patent, cumulative innovation, and optimal policy. International Journal of Industrial Organization 23, 535-562.

Furukawa, Y., 2007. The protection of intellectual property rights and endogenous growth: Is stronger always better?. Journal of Economic Dynamics & Control 31, 3644-3670.

Futagami, K., Iwaisako, T., 2007. Dynamic analysis of patent policy in an endogenous growth model. Journal of Economic Theory 132, 306-334.

Gallini, N.T., 2002. The economics of patents: Lessons from recent U.S. patent reform. Journal of Economic Perspectives 16, 131-154.

Gancia, G., Zilibotti, F., 2005. Horizontal innovation in the theory of growth and development. In: Aghion, P, Durlauf, S.N. (Eds.), Handbook of Economic Growth, vol. 1A. Elsevier, Amsterdam, pp. 111-170.

Gilbert, R., Shapiro, C., 1990. Optimal patent length and breadth. RAND Journal of Economics 21, 106-112.

Goh, A.-T., Olivier, J., 2002. Optimal patent protection in a two-sector economy. International Economic Review 43, 1191-1214.

Horii, R., Iwaisako, T., 2007. Economic growth with imperfect protection of intellectual property rights. Journal of Economics 90, 45-85.

Horowitz, A.W., Lai, E.L.-C., 1996. Patent length and the rate of innovation. International Economic Review 37, 785-801.

Iwaisako, T., Futagami, K., 2003. Patent policy in an endogenous growth model. Journal of Economics 78, 239-258.

Judd, K.L., 1985. On the performance of patents. Econometrica 53, 567-585.

Klemperer, P., 1990. How broad should the scope of patent protection be?. RAND Journal of Economics 21, 113-130.

Koléda, G., 2004. Patents' novelty requirement and endogenous growth. Revue d'Économie Politique 114, 201-221.

Krugman, P., 1979. A model of innovation, technology transfer, and the world distribution of income. Journal of Political Economy 87, 253-266.

Kwan, Y.K., Lai, E.L.-C., 2003. Intellectual property rights protection and endogenous economic growth. Journal of Economic Dynamics & Control 27, 853-873.

Lemley, M.A., Shapiro, C., 2005. Probabilistic patents. Journal of Economic Perspectives 19, 75-98.

Levin, J.J., 1959. On the matrix Riccati equation. Proceedings of the American Mathematical Society 10, 519-524.

Li, C.-W., 2001. On the policy implications of endogenous technological progress. Economic Journal 111, C164-C179.

Rivera-Batiz, L.A., Romer, P.M., 1991. Economic integration and endogenous growth. Quarterly Journal of Economics 106, 531-555.

Romer, P.M., 1990. Endogenous technological change. Journal of Political Economy 98, S71-S102.

Tandon, P., 1982. Optimal patents with compulsory licensing. Journal of Political Economy 90, 470-486.