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Equivalence between the core and the set of competitive allocations is shown to hold in the limit for large coalition production economies that allow for both decreasing returns to scale and superadditivity of the technology correspondence. Along the way, we shed some light on a separation argument problem with the alleged proof in Böhm (1974a), and on the unfortunate stringency—in terms of an implied near absolute degree of uniformity of ideas or production capabilities across society—of Arrow and Hahn's (1971) assumptions to tackle core equivalence. Our approach both amends the existing proofs in the literature and widens the scope of the Core Equivalence Theorem, as we show not only by weakening its hypotheses but also by proposing an extension of this classic result to a fuzzy coalition production economy a la Aubin (1979).

Keywords: General Equilibrium, Core Equivalence, Coalition Production Economy, Edgeworth's Conjecture, Fuzzy Coalition

JEL Codes: D50, D71, C71

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1 Introduction

Edgeworth's (1881) conjecture of (what is nowadays called) *core equivalence*—the idea that "it is not (much) more constraining to select only outcomes permitted by the market mechanism than to permit any core allocation" (Anderson, 1994)—can be regarded as a basic test of perfect competition (Anderson and Zame, 1997; Gretsky et al., 1999).[1] In fact, it is arguably so basic as to merit the "Third Fundamental Theorem of Welfare Economics" title (see Aliprantis and Burkinshaw, 1991).

A proof of this conjecture was first proposed by Edgeworth (1881) himself—with the aid of some differential calculus and illustrations—in the context of economies with two goods and two types of individuals replicated to infinity, and then fourscore years later for economies containing an arbitrary, albeit finite, number of commodities and individuals by Scarf (1961) and Debreu and Scarf (1963). More general enlargement processes for type economies have also been considered and, despite not producing the same sort of clear-cut equivalence result, also (typically) lead to approximate decentralization of core elements according to different convergence notions, an analysis introduced in Hildenbrand (1970), strengthened in Arrow and Hahn (1971) and Anderson (1978), and thoroughly surveyed in Anderson (1992, sec. 5).

The hypothesis of finiteness of the number of goods has been generalized in Peleg and Yaari (1970) and in Aliprantis et al. (1987a) (among others), where consumption sets are assumed to be contained in $\mathbb{R}^{\mathbb{N}}$ and in abstract vector lattices, respectively. A significant portion of the literature also dispenses altogether with finiteness of the very set of individuals, by appealing—as justified in Hildenbrand (1970)—to "the convenient, if somewhat artificial device, of an economy with a continuum of agents," (Hildenbrand and Kirman, 1976, sec. 5.8) as in Aumann (1964) (see also Vind, 1964; Richter, 1971). Actually, the core equivalence result does not break down even in an economy with a "not-too-atomistic" measure space of agents (Gabszewicz and Mertens, 1971; Shitovitz,

[1] In Edgeworth's (1881, pp. 38–39) original words—to be picked up again in the next section—,

> we see that in general for any number short of the practically infinite (if such a term be allowed) there is a finite length of contract-curve [...] at any point of which if the system is placed, it cannot by contract or recontract be displaced; that there are an indefinite number of final settlements, a quantity continually diminishing as we approach a perfect market.

1973)—an analysis that unfortunately does not apply to the counting measure in a replica economy as first envisaged by Edgeworth.[2]

Incorporating production, however, introduces new challenges. First and foremost, one must answer a fundamental ontological question on the nature of the firm: is it an exogenously given productive enterprise, or simply a collection of more basic economic agents (individuals)? The former interpretation leads to the consideration of private ownership economies, in which individuals are endowed not only with commodities but also with shares in each firm, and for which the issue of core equivalence has been duly addressed in Aliprantis et al. (1987b) and later, with a more elaborate (taking both corporate governance and feasibility constraints into account) notion of the core, in Xiong and Zheng (2007).[3] Such analyses make thorough—and rightful—use of the underlying additivities of the standard Arrow-Debreu model.

The latter interpretation gives rise to coalition production economies, in which sets of individuals (coalitions) are endowed with ideas/production possibilities, and decide to join efforts in goods production at their own discretion, as in Hildenbrand (1970, 1974). However, these works did not aim to dispense altogether with additivities in the way productive ideas can be aggregated, making the considered economies akin to simple private ownership economies in which each agent is the sole owner of a firm, as argued in Oddou (1982). In contrast, additivity of the original economy's technology correspondence is not imposed here: our model allows for the whole (the productive capacity of any nonempty coalition) to be greater than the sum of its parts (the productive capacities of the elements of any of its partitions), as in Arrow and Hahn (1971, ch. 8) and Böhm (1974a). As a result, equilibrium dividends are not necessarily determined simply through Shapley values (for more on this point, see Sondermann, 1974).[4]

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[2]For a detailed account of the findings of some of these and many other works within this branch of the literature, see Anderson (1992).

[3]See also Turchick, Xiong, and Zheng (2022).

[4]Nevertheless, it should be clear to the reader that, just like Hildenbrand's additive model, the present one has nothing to say about which firms form in equilibrium. In fact, without loss of generality, one can assume that the only operating firm is the grand coalition. Cf. Ichiishi (1977), who addresses this issue for balanced—not necessarily superadditive—economies.

Moreover, not even inter-replica additivity of the economy is assumed—so that, for instance, given a base economy with a goalkeeper and an outfielder only, the "production possibilities" of a coalition composed of ten clone outfield players and one goalkeeper are allowed to be greater than those of ten smaller ensembles, one of them consisting of an outfielder and a goalie, while the other nine, of an outfielder only. This is in contrast to the private ownership environment, for which the production possibilities of the replica economy are just a scaled-up version of the production possibilities of the original economy—thus hindering, in the previous example, the prospects for a functional team of eleven players, with each outfielder performing a specialized task on the pitch, being formed even in the tenth replica. Therefore, not only are firms and shares not given exogenously in our enlarging sequences of economies, but also the possibility for positive productive externalities is as broad as it can be.

One should resist the temptation of trying to argue at this point that validity of Edgeworth's conjecture in the presence of the mentioned inter-replica externalities follows in a more or less straightforward fashion from its validity in an additive framework, such as the private ownership one. In fact, when compared to Example 2 in Appendix A, Example 3 there shows that, although these externalities might, indeed, hasten shrinkage (in the usual sense) of the core throughout the replication process, they also subject to shrinkage the very set of competitive equilibrium allocations.

In addition to this fundamental ontological question on the nature of the firm, another primary distinction arising from the consideration of production appears in the definition of core, which hinges upon the definition of blocking. Which resources may be employed by a coalition when considering the possibility of blocking (i.e., objecting to / improving upon) a given feasible allocation? Although there is no clear-cut answer in the case of private ownership economies (see Xiong and Zheng, 2007, 2008; Kreps, 2013, sec. 15.3), the answer in the case of coalition production economies leaves less room for dispute: everything with which the coalition is endowed in the aggregate, not only in terms of commodities (which naturally aggregate in an additive fashion) but also in terms of productive ideas.

As in the case of pure exchange economies, approaches to the Core Equivalence Theorem for production economies also vary in their strategies for the modeling of large economies. In atomless economies, equivalence between the limiting core and the competitive equilibrium allocations has been established, in the case of a finite number of commodities, both for technology correspondences that are additive (see Hildenbrand, 1968, and Cornwall, 1969) and superadditive (see Oddou, 1976). It has also been established in the case of an infinite number of goods in Bewley (1973), albeit under the additional assumption that production sets exhibit constant returns to scale.

It would only be natural to expect this "Third Welfare Theorem" to also hold in productive economies without resorting to the continuum agents artifice. Indeed, decentralization of limiting core allocations of replica economies was also achieved by Debreu and Scarf (1963) under constant returns to scale—ergo, zero profits—, at least in the special case of publicly accessible technologies.[5] However, when the possibility of decreasing returns to scale enters the picture, it becomes imperative to specify which dividend distributions are to be allowed in a competitive equilibrium and which ones are not, as pointed out by Sondermann (1974). The significance of this point is further illustrated by Example 1 in our Appendix A.

Determination of individual dividends is not an issue if the technology correspondence is additive. This is due to the coincidence between the unique imputation of a TU game and its Shapley value, a point to which we return in the next section. It is also not an issue even under superadditivity of the technology correspondence if, as imposed by Champsaur (1974), the production set of the whole economy happens to present constant returns to scale, thus implying zero profits in equilibrium.

Still in the more general superadditive case, Arrow and Hahn (1971) were able to obtain core equivalence for type economies via the ingenious, elegant trick of envisaging individuals as marketable productive resources. By doing so, the original economy can be associated with one with publicly accessible and constant returns technology—one for which the separation argument of Debreu and Scarf (1963) can then be applied.[6] However,

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[5]This is, in fact, a trivial instance of additivity, in that the sum of a convex cone with itself is equal to itself. Liu and Liu (2014) obtain the same result without the public access assumption, albeit sticking to the constant returns one.

[6]Tricks of the same sort are also employed in the literature to deal with other issues (e.g., Bonnisseau, del Mercato, and Siconolfi (2023)).

despite their attempt to abstain from imposing public access to a common technology, their setup tacitly requires an asymptotic version of public access: the production plans available to society as a whole regardless of the scale of production could also be carried out by any individual by him/herself. The aforementioned constraints—explicit and implicit—are greatly relaxed in our approach.

A first reading of Böhm (1974a) suggests the overcoming of such a difficulty. Further scrutiny, however, reveals that the hypothesis of a constant returns aggregate production set, although not made explicit, still underlies the separating hyperplane argument proposed there. In fact, we show by means of an example that that proof might break down under strictly decreasing returns to scale.

In order to get these difficulties dealt with once and for all, we return to the most basic model of a growing finite coalition production economy with superadditivity and show how the Core Equivalence Theorem can be obtained once both Debreu and Scarf's (1963) argument and Arrow and Hahn's (1971) trick are adapted and combined.

In addition to reassuring the reader that the result of Böhm (1974a) is essentially correct—so that also other works that use it, such as Greenberg (1979), do not have to go through renewed due diligence—, we show that the same proof strategy can also be applied to prove the Core Equivalence Theorem for fuzzy coalition production economies, in the tradition of Aubin (1979). In this sense, we see that Edgeworth was actually right even for small economies—as long as firms can be made up of fractions of individuals' productive capacities. Having passed this test (as argued by Anderson and Zame, 1997; Gretsky et al., 1999), the fuzzy version of the coalition production economy model with decreasing (or constant) returns to scale and superadditivity of the technology correspondence ultimately seems to also merit renewed methodological worth.

The remainder of the paper is organized as follows. Section 2 reviews the basic model, its key definitions, and its basic results. Section 3 examines the limitations of the existing theory. Building on this, Section 4 provides a proof for the Core Equivalence Theorem. Section 5 then presents its fuzzy extension, with Section 6 offering some concluding remarks.

2 Definitions and preliminary results

Our environment is that of coalition production economies, initially outlined by Hildenbrand (1968). In the following, it is defined along with the assumptions necessary for our analysis.

Definition 1. Let there be given a finite nonempty set of individuals $\mathcal{I}$, each $i \in \mathcal{I}$ with an endowment $\mathbf{e}^i \in \mathbb{R}^L_+$ and a rational preference relation (i.e., a total preorder) $\succsim^i$ on $\mathbb{R}^{\mathbb{L}}_+$, and a technology correspondence $Y : 2^{\mathcal{I}} \rightrightarrows \mathbb{R}^L$ mapping each subset of $\mathcal{I}$ (to be called a coalition of the economy) into its production possibility set.[7] We shall refer to $\mathcal{E} = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y\right)$ as a *coalition production economy* (or, for short, an *economy*) if the following assumptions hold:

A1. For all $i \in \mathcal{I}$, $\succsim^i$ is lower semicontinuous, strictly monotone (i.e., $\mathbf{x}^i \geq \mathbf{x}'^i$ implies $\mathbf{x}^i \succsim^i \mathbf{x}'^i$ and $\mathbf{x}^i \gg \mathbf{x}'^i$ implies $\mathbf{x}^i \succ^i \mathbf{x}'^i$), and convex, over the consumption set $\mathbb{R}^L_+$;[8]

A2. For all $S \in 2^{\mathcal{I}}$, $Y(S)$ is convex and contains $\mathbf{0}$ (and only $\mathbf{0}$, if $S = \varnothing$);

A3. Given two disjoint coalitions S and S', $Y(S \cup S') \supseteq Y(S) + Y(S')$, where the plus sign denotes Minkowski sum.

This definition clearly encompasses the case of publicly accessible technologies previously mentioned in the introduction and found in Debreu and Scarf (1963), where one must have $Y(S) = Z$ for all nonempty coalitions S, with Z being a convex cone.[9] Our approach, however, allows coalitions to have distinct production possibility sets, and not necessarily of the constant returns type (not even the production set of the grand coalition).

Additional degrees of freedom on the production side are provided by the superadditivity assumption A3: although business associates could certainly opt not to interact with

[7]Henceforth, the empty set is considered a coalition, as in Hildenbrand (1968), Cornwall (1969), Sondermann (1974), and Böhm (1974a).

[8]This convexity will be assumed to be strict in a technical assumption later on in this section. The reason it is not included here is a subtle one: we need a certain object ($\mathcal{E}'$) within the proof of the Core Equivalence Theorem to be an economy.

[9]In this paper, *cone* always refers to a cone containing the origin. It needs not be pointed—that is, reversibility is not an issue.

each other, doing so would not bring about new aggregate production possibilities—and, as explained shortly, there is no loss in generality in assuming away such individualistic behavior in equilibrium and just registering total production, as in Sondermann (1974), Böhm (1974a) and Oddou (1976). Coupled with A2, it immediately implies monotonicity (i.e., $S \subseteq S' \Rightarrow Y(S) \subseteq Y(S')$), and is clearly weaker than the supermodularity (or convexity) assumption proposed in Sondermann (1974) (according to which one must have $Y(S \cup S') + Y(S \cap S') \supseteq Y(S) + Y(S')$ even if S and S' are not disjoint).

In the different context of a continuum of individuals as in Aumann (1964), the assumption in Hildenbrand (1968, 1974) regarding how the production possibility set of a coalition relates in an exact fashion to the production possibility sets of smaller coalitions would here translate to additivity of Y: $Y(S) = \sum_{i \in S} Y(\{i\}), \forall S \in 2^{\mathcal{I}} \setminus \{\varnothing\}$. Such a simple structure for technology aggregation is evidently compatible with that suggested in Debreu and Scarf (1963), where each $Y(\{i\})$ is the same convex cone Z, so that $\sum_{i \in S} Y(\{i\})$, as the convex hull of $\bigcup_{i \in S} Y(\{i\})$ $(= Z)$ is simply Z itself, regardless of the coalition S under consideration.

Since slight variations of the following definitions can be found in the literature, we ought to present them in full. Given an economy $\mathcal{E} = \left(\mathcal{I}, (\mathbf{e}^i)_{i \in \mathcal{I}}, (\succsim^i)_{i \in \mathcal{I}}, Y\right)$, an *allocation* of $\mathcal{E}$ will be described simply by the consumption bundle it assigns to each individual—in other words, it is an element of $\mathbb{R}_+^{L|\mathcal{I}|}$. Given a nonempty coalition S, we call an allocation $\mathbf{x} = (\mathbf{x}^i)_{i \in \mathcal{I}}$ *S-feasible* if $\sum_{i \in S} (\mathbf{x}^i - \mathbf{e}^i) \in Y(S)$ (and just *feasible* if it is $\mathcal{I}$-feasible). A feasible allocation $(\mathbf{x}^i)_{i \in \mathcal{I}}$ is *blocked* by a nonempty coalition S if there exists an S-feasible allocation $\mathbf{x}' = (\mathbf{x}'^i)_{i \in \mathcal{I}}$ such that $\mathbf{x}'^i \succ^i \mathbf{x}^i$ for all $i \in S$.[10,11]

Definition 2. The *core* of $\mathcal{E}$ is the set $C(\mathcal{E})$ of all feasible allocations of $\mathcal{E}$ that are not

[10]By A2, without loss of generality, $\mathbf{x}'$ can also be required to be feasible, or both S- and $(\mathcal{I} \setminus S)$-feasible (simply take $\mathbf{x}'^i = \mathbf{e}^i$ for each $i \in \mathcal{I} \setminus S$). Thus, in this work, a differentiation between "inconsiderate" and "considerate" blocking, as in Xiong and Zheng (2007), would result immaterial.

[11]This strong notion of blocking can be seen to be equivalent to weak blocking (replacing "$\mathbf{x}'^i \succ^i \mathbf{x}^i$ for all $i \in S$" by "$\mathbf{x}'^i \succsim^i \mathbf{x}^i$ for all $i \in S$ with at least one preference strict") once technical assumption TA1 is made. However, at this point, such an equivalence is not guaranteed. For instance, in an exchange economy with aggregate endowment of two units of each of two goods, individual A caring only about good x_1, and individual B only about x_2, $\left(\mathbf{x}^A, \mathbf{x}^B\right) = ((1,0),(1,2))$ is weakly blocked by $\{A, B\}$ (consider $\left(\mathbf{x}'^A, \mathbf{x}'^B\right) = ((2,0),(0,2))$), but is not strongly blocked by that coalition.

blocked by any coalition of that economy.[12]

We now proceed to define the competitive equilibria of $\mathcal{E}$. It is useful to first establish a bit of notation: given $\mathbf{p} \in \mathbb{R}^L_{++}$, the *potential profit function* $\Pi_{\mathcal{E},\mathbf{p}} : 2^{\mathcal{I}} \to [0, \infty]$ is given by $\Pi_{\mathcal{E},\mathbf{p}}(S) = \sup_{\boldsymbol{v} \in Y(S)} \mathbf{p} \cdot \boldsymbol{v}$ (where the supremum of an unbounded-from-above subset of $\mathbb{R}$ is defined as $+\infty$, and the possibility of inaction of each coalition S guarantees $\Pi_{\mathcal{E},\mathbf{p}} \geq 0$).

Definition 3. Given an economy $\mathcal{E} = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y\right)$, a *Walras (or competitive) equilibrium* of $\mathcal{E}$ is a tuple $\left((\mathbf{x}^i)_{i\in\mathcal{I}}, (t^i)_{i\in\mathcal{I}}, \mathbf{p}, \mathbf{y}\right) \in \mathbb{R}^{L|\mathcal{I}|}_+ \times \mathbb{R}^{|\mathcal{I}|}_+ \times \mathbb{R}^L_{++} \times Y(\mathcal{I})$ such that:

E1. $\sum_{i\in\mathcal{I}} t^i \leq \mathbf{p} \cdot \mathbf{y}$;

E2. For all $S \in 2^{\mathcal{I}}$, $\sum_{i\in S} t^i \geq \Pi_{\mathcal{E},\mathbf{p}}(S)$;

E3. For all $i \in \mathcal{I}$, $\mathbf{x}^i$ is a maximal element of $\left\{\mathbf{c}^i \in \mathbb{R}^L_+ : \mathbf{p} \cdot \mathbf{c}^i \leq \mathbf{p} \cdot \mathbf{e}^i + t^i\right\}$ with respect to $\succsim^i$; and

E4. $\sum_{i\in\mathcal{I}} \mathbf{x}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i + \mathbf{y}$.

In this case, we refer to $(\mathbf{x}^i)_{i\in\mathcal{I}}$ as a *Walras equilibrium allocation* of $\mathcal{E}$.

As usual, we denote by $W(\mathcal{E})$ the set of all Walras equilibrium allocations of $\mathcal{E}$. This definition aligns perfectly with the classic literature on coalition production economies, including Hildenbrand (1968, 1974), Böhm (1974a), Sondermann (1974), and Oddou (1976). Evidently, it also generalizes the definition in Debreu and Scarf (1963) (for which one necessarily obtains $\Pi_{\mathcal{E},\mathbf{p}} \equiv 0$ and each $t^i = 0$ in equilibrium).

It is worth noting that only the aggregate production plan $\mathbf{y} \in Y(\mathcal{I})$ enters the above definition. The issue of which firms form in equilibrium and what each one of them decides to take as input and produce as output is rendered moot by the superadditivity of Y, in the sense that any aggregate production obtained with any configuration of firms in

[12] If, in addition to the assumptions above, each $\succsim^i$ is upper semicontinuous, Y is supermodular, and at least the asymptotic cone of $Y(\mathcal{I})$ has the no free lunch property, then $C(\mathcal{E}) \neq \varnothing$ (see (Böhm, 1974b; Champsaur, 1974)).

equilibrium could be mimicked by the grand coalition $\mathcal{I}$.[13]

It may also be noted that, given the present local nonsatiation context (implied by the strict monotonicity of preferences in A1), one could get rid of E1 altogether, since it follows from E3 and E4: $\sum_{i\in\mathcal{I}} t^i = \sum_{i\in\mathcal{I}} \mathbf{p}\cdot(\mathbf{x}^i - \mathbf{e}^i) = \mathbf{p}\cdot\sum_{i\in\mathcal{I}}(\mathbf{x}^i - \mathbf{e}^i) = \mathbf{p}\cdot\mathbf{y}$.

To understand the game-theoretic motivation behind the profit distribution vector $(t^i)_{i\in\mathcal{I}}$, note that given a price vector $\mathbf{p}$, as long as $\Pi_{\mathcal{E},\mathbf{p}}(\mathcal{I}) < \infty$, $(\mathcal{I}, \Pi_{\mathcal{E},\mathbf{p}})$ is a cooperative game with transferable utility, for which E1 and E2 simply say that $(t^i)_{i\in\mathcal{I}}$ belongs to its core. Intuitively, at equilibrium, individuals have no incentive to coalesce and form new firms, as it would not yield higher profits.

There are, of course, alternative sets of payoff vectors one might consider for the transferable utility game $(\mathcal{I}, \Pi_{\mathcal{E},\mathbf{p}})$. In principle, these could be highly restrictive, containing for instance only the Shapley value, all the way up to much larger sets, such as that containing every imputation of the game. In the case of an additive production correspondence, as in Hildenbrand (1968, 1974), this choice becomes inconsequential: all these sets collapse, resulting in $t^i = \Pi_{\mathcal{E},\mathbf{p}}(\{i\})$ for every individual i.[14] In contrast, under superadditivity, choosing the imputation set—so that the equilibrium profit distribution is only required to be efficient and individually rational—instead of the core may invalidate the First Welfare Theorem's direction of core equivalence (for further details, see Example 1 in Appendix A).[15]

[13]Formally, if one wishes to follow Ichiishi's (1977) approach and define a competitive equilibrium as a tuple $\left((\mathbf{x}^i)_{i\in\mathcal{I}}, (t^i)_{i\in\mathcal{I}}, \mathbf{p}, Q, (\mathbf{y}^j)_{j\in Q}\right)$, where Q is a partition of $\mathcal{I}$ representing the firms that are formed and $\mathbf{y}^j \in Y(j)$ is the production plan of firm $j \in Q$ (with conditions E1 and E4 duly adjusted to read "$\sum_{i\in j} t^i \leq \mathbf{p}\cdot\mathbf{y}^j, \forall j \in Q$" and "$\sum_{i\in\mathcal{I}} \mathbf{x}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in Q} \mathbf{y}^j$"), then no "new" (in terms of individual consumption bundles) $\left((\mathbf{x}^i)_{i\in\mathcal{I}}, Q, (\mathbf{y}^j)_{j\in Q}\right)$-type allocations would emerge in equilibrium. In fact, if $\left((\mathbf{x}^i)_{i\in\mathcal{I}}, (t^i)_{i\in\mathcal{I}}, \mathbf{p}, Q, (\mathbf{y}^j)_{j\in Q}\right)$ is an equilibrium, then so is $\left((\mathbf{x}^i)_{i\in\mathcal{I}}, (t^i)_{i\in\mathcal{I}}, \mathbf{p}, \{\mathcal{I}\}, \mathbf{y}^{\mathcal{I}}\right)$, as can be checked by summing in j the inequalities given in E1 and noting that $\sum_{j\in Q} \mathbf{y}^j \in \sum_{j\in Q} Y(j) \subseteq Y\left(\bigcup_{j\in Q} j\right) = Y(\mathcal{I})$ by Y's superadditivity. Hence, the lack of loss of generality in our treating the grand coalition as the economy's sole firm in equilibrium, and our focus on the consumption entries of allocations.

[14]For other possibilities, see the "core catchers" in Branzei, Dimitrov, and Tijs (2008, ch. 2).

[15]We gratefully acknowledge Andrés Carvajal's and Charles Zheng's remarks on an earlier version of this work that led us to consider alternative possibilities for E2.

If, instead of assuming superadditivity, we adopted the supermodularity assumption, then the game $(\mathcal{I}, \Pi_{\mathcal{E},\mathbf{p}})$ would be convex, implying, as shown by Shapley (1971), nonemptiness of its core. Additionally, as Sondermann (1974) demonstrated, the set of Walrasian equilibrium allocations of $\mathcal{E}$ would also be nonempty.

Definition 4 below presents the fundamental mathematical object handled by the Core Equivalence Theorem: a sequence of replica economies. Given the set of individuals $\mathcal{I}$ of an economy and any $r \in \mathbb{N}$, let us establish the notation $\mathcal{I}_r := \mathcal{I} \times \{1, \ldots, r\}$, so that members of $\mathcal{I}_r$ have not only any of the names (or types) from $\mathcal{I}$, but also any of the "surnames" 1, ..., r.[16] For each $S \in 2^{\mathcal{I}_r}$, the *profile* of S is defined as the $|\mathcal{I}|$-tuple $\kappa(S) = (\kappa^i(S))_{i \in \mathcal{I}}$, where $\kappa^i(S)$ represents the number of members of S named i. Henceforth, we write I for $|\mathcal{I}|$.

Definition 4. A sequence $(\mathcal{E}_r)_{r \in \mathbb{N}}$ is a *sequence of replica economies* of the economy $\mathcal{E} = \left(\mathcal{I}, (\mathbf{e}^i)_{i \in \mathcal{I}}, (\succsim^i)_{i \in \mathcal{I}}, Y\right)$ if each $\mathcal{E}_r = \left(\mathcal{I}_r, (\mathbf{e}^i)_{(i,q) \in \mathcal{I}_r}, (\succsim^i)_{(i,q) \in \mathcal{I}_r}, Y_r\right)$ is an economy, where the technology correspondences $Y_r : 2^{\mathcal{I}_r} \rightrightarrows \mathbb{R}^L$ satisfy the following assumptions:

A4. $Y_1(S \times \{1\}) = Y(S), \forall S \in 2^{\mathcal{I}}$;

A5. Given $r \in \mathbb{N}$ and $S \in 2^{\mathcal{I}_r}$, $Y_{r+1}(S) = Y_r(S)$;

A6. Let $r \in \mathbb{N}$ and $S, T \in 2^{\mathcal{I}_r}$. If there exists $n \in \mathbb{N}$ such that $n\kappa(S) = \kappa(T)$, then $nY_r(S) = Y_r(T)$.

Assumption A4 aligns the production possibilities of the original economy $\mathcal{E}$ with those of the first of its replica economies in the sequence $(\mathcal{E}_r)_{r \in \mathbb{N}}$ (i.e., $\mathcal{E}_1$), whereas assumption A5 aligns the production possibilities of a replica economy with any (by induction) subsequent economy in the same sequence of replica economies. The $n = 1$ case of Assumption A6 specifies the surname-invariance property, according to which the synergy gains resulting from individuals of a replica economy coalescing has nothing to do with the specific levels to which they belong.

The aforementioned properties ensure that the Y_r correspondences give rise to a well-defined correspondence $Y_{\mathbb{Z}} : \mathbb{Z}_+^I \rightrightarrows \mathbb{R}^L$ mapping each coalition profile to the production set of any coalition with that profile. Since the above definition stipulates that each $\mathcal{E}_r$

[16] Wherever $\mathbb{N}$ or "naturals" is mentioned in this work, 0 is not being included.

must be an economy, $Y_{\mathbb{Z}}$ inherits the properties that follow from each Y_r satisfying A2 and A3. That is, $Y_{\mathbb{Z}}$ is convex-valued, inaction allowing, satisfies $Y_{\mathbb{Z}}(\mathbf{0}) = \{\mathbf{0}\}$, and is superadditive:

$$Y_{\mathbb{Z}}(\mathbf{k}+\mathbf{k}') \supseteq Y_{\mathbb{Z}}(\mathbf{k}) + Y_{\mathbb{Z}}(\mathbf{k}'), \forall\, \mathbf{k}, \mathbf{k}' \in \mathbb{Z}_+^I.$$

An additional property of $Y_{\mathbb{Z}}$, resulting from Assumption A6, is its homogeneity:

$$Y_{\mathbb{Z}}(n\mathbf{k}) = nY_{\mathbb{Z}}(\mathbf{k}),\ \forall n \in \mathbb{N},\ \forall\, \mathbf{k} \in \mathbb{Z}_+^I.$$

Therefore, it is not the case that increasing returns to scale are forced into the replication process as a means to achieve the appropriate level of core shrinkage in the limit. On the contrary, the example in Böhm (1973) suggests how lack of homogeneity would impose a challenge to core equivalence.

These assumptions on the replication structure are equivalent, modulo continuity, to those of Arrow and Hahn (1971, p. 201). As in that work and in Böhm (1974a), the replication process is, on the one hand, general enough to allow for non-additivities across different levels of the replica economies. For instance, it may very well be the case that $Y_{\mathbb{Z}}(1,10) \supsetneqq Y_{\mathbb{Z}}(1,1) + 9Y_{\mathbb{Z}}(0,1)$ (a mathematical formulation of the soccer remark made in the introduction). The effects of such positive production externalities are illustrated in Example 3 in Appendix A. On the other hand, the homogeneity prescribed in Assumption A6 minimally secures equal treatment in the core of a replica economy (as argued in Lemma 1 ahead) and extensibility of $Y_{\mathbb{Z}}$ to a $\mathbb{Q}_+$-homogeneous correspondence $Y_{\mathbb{Q}} : \mathbb{Q}_+^I \rightrightarrows \mathbb{R}^L$: given $\boldsymbol{\chi} \in \mathbb{Q}_+^I$, just select any positive integer N such that $N\boldsymbol{\chi}$ results in an integer vector, and define $Y_{\mathbb{Q}}(\boldsymbol{\chi}) := (1/N)\, Y_{\mathbb{Z}}(N\boldsymbol{\chi})$. Claim 3 shows that there is no ambiguity in this definition, and that $Y_{\mathbb{Q}}$ thusly defined is also inaction allowing, convex-valued, and superadditive.

The existence of this homogeneous extension is key in Arrow and Hahn's (1971) ingenious approach to core equivalence—that of replacing the given economy with an extended Debreu and Scarf economy (one with a publicly accessible, constant returns technology). However, as shown in Section 3, an additional assumption in that work, leading to upper hemicontinuity of $Y_{\mathbb{Q}}$, substantially narrows down the scope of their analysis. Fortunately, the approach advanced in Section 4, similar in spirit to Arrow and Hahn's (1971)—but with an extra dose of constructiveness—secures the validity of core equivalence in the present, more general, environment.

Beyond our work and that of Arrow and Hahn (1971), homogeneity is also featured in other studies. A weaker form, applying only to coalitions with the same amount of each individual type, is used by Böhm (1974a). In the context of atomless economies, homogeneity is present not only in Hildenbrand (1968, 1974), where it emerges trivially from additivity and convex-valuedness of the technology correspondence, but also in Oddou (1976, 1982). In particular, Oddou (1982) revisits an example in Böhm (1973) to argue that core equivalence might fail without homogeneity.

We now introduce three technical assumptions that will be essential for the remainder of the text. These assumptions will be explicitly referenced whenever they are applied, unlike the basic assumptions A1–A6, which are embedded in the definitions. Given an economy $\mathcal{E} = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y\right)$ and a sequence $(\mathcal{E}_r)_{r\in\mathbb{N}}$ of replica economies of $\mathcal{E}$, consider the following:

TA1. For all $i \in \mathcal{I}$, $\succsim^i$ is strictly convex over the consumption set $\mathbb{R}^L_+$;

TA2. There is at least one $\widetilde{\mathbf{y}} \in Y(\mathcal{I})$ such that $\sum_{i\in\mathcal{I}} \mathbf{e}^i + \widetilde{\mathbf{y}} \gg \mathbf{0}$;

TA3. For all $r \in \mathbb{N}$ and all $S \in 2^{\mathcal{I}_r}$, $Y_r(S)$ is closed.

Assumption TA2 ensures the existence of at least one feasible allocation of the economy in which aggregate consumption of each good is positive. In conjunction with TA3, it will be used in the proof of theorem 2. Note that, when combined with A4, TA3 implies that Y, the technology correspondence of $\mathcal{E}$, is closed-valued.

Assumption TA1, once paired with the strict monotonicity of $\succsim^i$ given by A1, yields strong monotonicity (i.e., $\mathbf{x}^i \underset{\neq}{\geq} \mathbf{x}'^i \Rightarrow \mathbf{x}^i \succ^i \mathbf{x}'^i$). It will be applied in Lemma 1 ahead, along with conditions A4, A5 and A6, which, despite not being ubiquitous in the coalition production economy literature (particularly irrelevance of surnames in Böhm, 1974, and the existence of a $Y_{\mathbb{Z}}$ correspondence in Liu and Liu, 2014; Liu, 2017), are behind the following key result of Debreu and Scarf (1963), in the absence of which the Core Equivalence Theorem cannot even be stated in the present form.[17] In it, we use the repetition notation:

[17]If one adopts the special case of an additive technology correspondence as suggested within parentheses in assumption P3 in Liu and Liu (2014)—and not only for Y but also for $Y_{\mathbb{Z}}$—, since each $Y(\{i\})$ there must be a convex cone, one obtains for each coalition S of a replica economy $Y_{\mathbb{Z}}(\kappa(S)) = \sum_{(i,q)\in S} Y(\{i\})$, consistent with the definition of S-feasibility ("S-attainability") there.

if $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{I}}$ is an allocation of $\mathcal{E}$, then the r-fold repetition of $\mathbf{x}$, ${}^r\mathbf{x} = \left({}^r\mathbf{x}^{(i,q)}\right)_{(i,q)\in\mathcal{I}_r}$, is the allocation of $\mathcal{E}_r$ for which ${}^r\mathbf{x}^{(i,q)} = \mathbf{x}^i, \forall (i,q) \in \mathcal{I}_r$. Similarly, if $\mathbf{t} = (t^i)_{i\in\mathcal{I}}$ is a profit distribution vector for the base economy, then ${}^r\mathbf{t} = \left({}^r t^{(i,q)}\right)_{(i,q)\in\mathcal{I}_r}$ is such that ${}^r t^{(i,q)} = t^i, \forall (i,q) \in \mathcal{I}_r$.

Lemma 1 (Equal Treatment Lemma)**.** Given an economy $\mathcal{E}$ and a sequence $(\mathcal{E}_r)_{r\in\mathbb{N}}$ of replica economies of $\mathcal{E}$ satisfying TA1, if $\mathbf{x} \in C(\mathcal{E}_r)$ for some $r \in \mathbb{N}$, then $\mathbf{x} = {}^r\mathbf{z}$ for some allocation $\mathbf{z}$ of $\mathcal{E}$.

Proof. Let $\mathbf{x} \in C(\mathcal{E}_r)$. For each $i \in \mathcal{I}$, let l_i be the surname of the (or one of the) less fortunate individual among all named i, according to their consumption bundles collected in allocation $\mathbf{x}$ (i.e., $\mathbf{x}^{(i,q)} \succsim^i \mathbf{x}^{(i,l_i)}, \forall q \in \{1, \ldots r\}$). Consider the coalition $S = \{(i,l_i) \in \mathcal{I}_r : i \in \mathcal{I}\}$.

Consider now the allocation $\mathbf{x}' := {}^r\mathbf{z}$, where $\mathbf{z}^i := (1/r)\sum_{q=1}^r \mathbf{x}^{(i,q)}, \forall i \in \mathcal{I}$. Note that $r\sum_{i\in\mathcal{I}}(\mathbf{z}^i - \mathbf{e}^i) = \sum_{(i,q)\in\mathcal{I}_r}\left(\mathbf{x}^{(i,q)} - \mathbf{e}^i\right) \in Y_{\mathbb{Z}}(\kappa(\mathcal{I}_r))$, by feasibility in $\mathcal{E}_r$ of its core member $\mathbf{x}$. This implies that $\mathbf{x}'$ is S-feasible:

$$\sum_{(i,q)\in S}\left(\mathbf{x}'^{(i,q)} - \mathbf{e}^i\right) = \sum_{(i,q)\in S}\left(\mathbf{z}^i - \mathbf{e}^i\right) \in \frac{1}{r}Y_{\mathbb{Z}}(\kappa(\mathcal{I}_r)) = Y_{\mathbb{Z}}(\kappa(\mathcal{I})) = Y_{\mathbb{Z}}(\kappa(S)) = Y_r(S),$$

where we have used the homogeneity of $Y_{\mathbb{Z}}$ and the surname-invariance of Y_r (A5).

Moreover, for each $(i,q) \in S$, convexity of preferences implies $\mathbf{x}'^{(i,q)} = \mathbf{z}^i \succsim^i \mathbf{x}^{(i,l_i)} = \mathbf{x}^{(i,q)}$. Since $\mathbf{x} \in C(\mathcal{E}_r)$, it must then be the case that, for all $i \in \mathcal{I}$, $\mathbf{x}'^{(i,l_i)} \sim^i \mathbf{x}^{(i,l_i)}$, that is, $\mathbf{z}^i \sim^i \mathbf{x}^{(i,l_i)}$. Strict convexity of preferences then warrants $\mathbf{z}^i = \mathbf{x}^{(i,q)}, \forall q \in \{1, \ldots, r\}$, so that $\mathbf{x} = {}^r\mathbf{z}$. □

In other words, any given element of the core of $\mathcal{E}_r$ is simply an r-repetition of an allocation of the (core of, as can be easily checked) base economy $\mathcal{E}$.[18] Let $C_r(\mathcal{E})$ be this subset of $\mathbb{R}_+^{LI}$ identifying $C(\mathcal{E}_r)$—that is, $\mathbf{z} \in C_r(\mathcal{E})$ if and only if ${}^r\mathbf{z} \in C(\mathcal{E}_r)$. The resulting $2^{(\mathbb{R}^{LI})}$-valued sequence $(C_r(\mathcal{E}))_{r\in\mathbb{N}}$ has the well-known shrinkage property, perfectly in line with Edgeworth's (1881) original claim that "there are an indefinite

[18]In the private ownership environment discussed in Xiong and Zheng, 2007, 2008, the equal treatment property for firms is required to address the blocking notions introduced by them. To establish this property, it is necessary to assume that the production possibility frontier for each firm is strictly concave.

number of final settlements, a quantity continually diminishing as we approach a perfect market."

Lemma 2. Given a sequence of replica economies $(\mathcal{E}_r)_{r\in\mathbb{N}}$ of an economy $\mathcal{E}$, the corresponding sequence $(C_r(\mathcal{E}))_{r\in\mathbb{N}}$ is a decreasing chain with respect to the partial order of inclusion in $\mathbb{R}^{LI}$.

Proof. We first check the more standard core shrinkage property. Fixed any $r \in \mathbb{N}$, if $\mathbf{x} \in C_{r+1}(\mathcal{E})$, then ${}^{r+1}\mathbf{x} \in C(\mathcal{E}_{r+1})$, so that ${}^{r+1}\mathbf{x}$ is feasible in $\mathcal{E}_{r+1}$. That is, $\sum_{(i,q)\in\mathcal{I}_{r+1}} (\mathbf{x}^i - \mathbf{e}^i) \in Y_{\mathbb{Z}}(\kappa(\mathcal{I}_{r+1}))$, or still, $(r+1)\sum_{i\in\mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i) \in (r+1)\, Y_{\mathbb{Z}}(\kappa(\mathcal{I}))$, by homogeneity of $Y_{\mathbb{Z}}$. Multiplying through by $r/(r+1)$ gives $r\sum_{i\in\mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i) \in rY_{\mathbb{Z}}(\kappa(\mathcal{I}))$, so that $\sum_{(i,q)\in\mathcal{I}_r} (\mathbf{x}^i - \mathbf{e}^i) \in Y_{\mathbb{Z}}(\kappa(\mathcal{I}_r)) = Y_r(\mathcal{I}_r)$—that is, ${}^{r}\mathbf{x}$ is feasible in $\mathcal{E}_r$. Now, if $S \in 2^{\mathcal{I}_r} \setminus \{\varnothing\}$ blocks ${}^{r}\mathbf{x}$ in $\mathcal{E}_r$, since its production possibility set is the same regardless of it being treated as a coalition of $\mathcal{E}_r$ or of $\mathcal{E}_{r+1}$, S also blocks ${}^{r+1}\mathbf{x}$ in $\mathcal{E}_{r+1}$, which is impossible. Therefore, ${}^{r}\mathbf{x} \in C(\mathcal{E}_r)$, so that $\mathbf{x} \in C_r(\mathcal{E})$. □

Following Aliprantis, Brown, and Burkinshaw (1987), we have the following

Definition 5. Given an economy $\mathcal{E}$ and a sequence of replica economies of $\mathcal{E}$, $(\mathcal{E}_r)_{r\in\mathbb{N}}$, an *Edgeworth equilibrium allocation* of $\mathcal{E}$ is an element of $\bigcap_{r=1}^{\infty} C_r(\mathcal{E})$.

Having introduced the main ingredients of the model, we can now turn to the central concept of our project: the Core Equivalence Theorem. This theorem states that the set of core allocations converges to the set of competitive equilibria, that is, Edgeworth equilibria can be rationalized by a decentralized system of prices.

In Section 4, we state this theorem and provide detailed proofs for each of its directions. But, prior to delving into the proofs, the following section will examine the limitations and shortcomings associated with the existing body of literature's attempts at establishing core equivalence within the finite coalition production economies framework.

3 Gaps in the literature

Before going into the proof of core equivalence, it is certainly worthwhile to get a better grasp of the extent to which the current theory of core equivalence in replica coalition production economies with decreasing returns to scale in the aggregate is limited. As stated in the introductory section, such an analysis has been provided in two different works: Böhm (1974a) and Arrow and Hahn (1971).

In order to better appreciate the critical flaw contained in the proof proposed in Böhm (1974a), let us recollect the main construction there. Given an economy $\mathcal{E} = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y\right)$ and a core allocation $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{I}}$, consider, for each $i \in \mathcal{I}$, the sets $\Gamma^i(\mathbf{x}) := \{\mathbf{z} \in \mathbb{R}^L : \mathbf{z} + \mathbf{e}^i \succ^i \mathbf{x}^i\}$. These were used as building blocks for $\Gamma^{\mathrm{DS}}(\mathbf{x}) := \mathrm{co}\left(\bigcup_{i\in\mathcal{I}} \Gamma^i(\mathbf{x})\right)$ in Debreu and Scarf (1963), whereas for the different (not necessarily a subset or a superset of $\Gamma^{\mathrm{DS}}(\mathbf{x})$) nonempty convex set $\Gamma^{\mathrm{B}}(\mathbf{x}) := \mathrm{co}\left(\bigcup_{(k^i)_{i\in\mathcal{I}}\in\mathbb{N}^I} \sum_{i\in\mathcal{I}} k^i\Gamma^i(\mathbf{x})\right)$ in Böhm (1974a).[19] Those works then manage to argue for disjointness between their respective $\Gamma(\mathbf{x})$ set and $Y(\mathcal{I})$, and to then confirm Edgeworth's Conjecture from the alleged existence of a vector hyperplane (i.e., one containing $\mathbf{0} \in \mathbb{R}^L$) separating those two sets.

The passing-through-the-origin property of such a hyperplane is not stressed in Debreu and Scarf (1963) for not being necessary in that context: a conical $Y(\mathcal{I})$ always makes such a construction possible (see theorems 11.3 and 11.7 in Rockafellar, 1970). However, it was apparently taken for granted in Böhm (1974a, p. 147, "$\mathbf{p}\cdot\mathbf{y} \leq 0$" on line 7), regardless of the possibility that $Y(\mathcal{I})$ is equipped with strictly decreasing returns there—otherwise, the issue of profit distribution stability in the definition of competitive equilibrium would not arise, given E1 (his equilibrium condition 4). This seemingly minor slip unfortunately contaminates the remainder of the proof of Edgeworth's Conjecture there, which relies on the presumption that the total profit of the economy is zero.

To see why it is not without loss of generality that such a property may be assumed, consider the economy $\mathcal{E} = \left(\{A\}, (\mathbf{e}^A), (\succsim^A), Y\right)$, where: $\mathbf{e}^A = (4,1)$, $\succsim^A$ is represented by the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ given by $u(x_1,x_2) = \sqrt{x_1} + \sqrt{x_2}$, $Y(\varnothing) = \{(0,0)\}$, and $Y(\{A\}) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \leq 0 \text{ and } x_2 \leq 2\sqrt{-x_1}\}$. For having just one individual, $\mathcal{E}$ can give rise to only one sequence of replica economies: $\boldsymbol{\mathcal{E}} = (\mathcal{E}_r)_{r\in\mathbb{N}}$, where, for all $r \in \mathbb{N}$, $\mathcal{E}_r = \left(\{A\}\times\{1,\ldots,r\}, (\mathbf{e}^A)_{q\in\{1,\ldots,r\}}, (\succsim^A)_{q\in\{1,\ldots,r\}}, Y_r\right)$, with $Y_r : 2^{\{A\}\times\{1,\ldots,r\}} \rightrightarrows \mathbb{R}^2$ being defined, in accordance with assumption A6—or, in Böhm's (1974a) notation, P2—by $Y_r(S) = |S|\, Y_r(\{A\})$.

[19] The reader is advised to recall the definition of $\mathbb{N}$ employed in this work.

As shown in Claim Y, $\mathbf{x} := (3,3) \in \bigcap_{r=1}^{\infty} C_r(\mathcal{E})$. It is certainly possible, following Böhm's (1974a, lemma 2) disjointness argument, to separate $Y(\{A\})$ from $\Gamma^{\mathrm{B}}(\mathbf{x})$ with an affine hyperplane. Nevertheless, because the conical hull of $Y(\{A\})$ (in this particular example, $\mathbb{R}_- \times \mathbb{R}$) intersects $\Gamma^{\mathrm{B}}(\mathbf{x})$ (e.g., both sets contain $(-1,3)$, since $(-1,3) + \mathbf{e}^A = (3,4) \succsim^A (3,3)$), it is impossible to separate $Y(\{A\})$ from $\Gamma^{\mathrm{B}}(\mathbf{x})$ by means of a vector hyperplane, as implicit in Böhm's (1974a) argument.

In the following section, we address this point by adopting Arrow and Hahn's (1971) ingenious approach of constructing an auxiliary economy where the total profit indeed equals zero. This adjustment enables us to find a separating hyperplane passing through the origin (of a space with some extra dimensions). It should be pointed out that our proof does not directly validate his theorem, as his assumptions do not include homogeneity of the technology correspondence, crucial for the construction of the mentioned auxiliary economy.

With this consideration in mind, one might ask if the findings presented in Arrow and Hahn (1971) could serve as a basis for deriving the Core Equivalence Theorem as a corollary. Adapting Arrow and Hahn's (1971) analysis (a "replica" core equivalence as a corollary to a more general "type" core equivalence result, notably theorem 6 on page 205) to the context of core equivalence defined in this study, despite the differences in the environments, is indeed feasible. However, this adaptation hinges on the assumptions made in their work, which we claim to be quite restrictive , not only for our environment but also for theirs.

Assumption 3 (p. 201) in Arrow and Hahn (1971) imposes the existence of a continuous extension of the $Y_{\mathbb{Q}}$ correspondence (defined in Section 2) to $\mathbb{R}_+^I$, which we denote $Y_{\mathbb{R}}$. To illustrate the potential restrictiveness of this assumption, let us just note that it would imply (given their assumptions 1 and 4) $\mathrm{A}\, Y(\mathcal{I} \setminus \{i\}) \subseteq Y(\{i\})$ for all $i \in \mathcal{I}$, where A is the recession (or asymptotic) cone operator. In other words, if any individual of the base economy $\mathcal{E}$ is singled out, he/she knows how to produce by him/herself absolutely any production plan that, irrespective of the scale of production, the whole remainder of society jointly knows how to produce—an unlikely scenario.[20]

[20]Words of caution about technical difficulties of this sort had already been offered by Oddou (1982, p. 7): "They [Arrow and Hahn] pose the problem of how to extend this correspondence to the set of all

Just to fix ideas, let us take $\mathcal{I} = \{1, \ldots, I\}$ (with individuals appearing in coalition profiles in their natural order), and argue that $\mathrm{A}\,Y(\mathcal{I} \setminus \{1\}) \subseteq Y(\{1\})$. If $\mathbf{y} \in \mathrm{A}\,Y(\mathcal{I} \setminus \{1\})$, then, due to $\mathrm{A}\,Y(\mathcal{I} \setminus \{1\})$ being a cone, we also have, for all $n \in \mathbb{N}$,

$$\begin{aligned} n\mathbf{y} &\in \mathrm{A}\,Y(\mathcal{I} \setminus \{1\}) = \mathrm{A}\,Y_{\mathbb{Z}}(0, 1, \ldots, 1) \subseteq Y_{\mathbb{Z}}(0, 1, \ldots, 1) \\ &\subseteq Y_{\mathbb{Z}}(0, 1, \ldots, 1) + Y_{\mathbb{Z}}(n, 0, \ldots, 0) \subseteq Y_{\mathbb{Z}}(n, 1, \ldots, 1) = Y_{\mathbb{R}}(n, 1, \ldots, 1), \end{aligned}$$

where the first set inclusion comes from the fact that $Y_{\mathbb{Z}}(0, 1, \ldots, 1)$ is a convex and closed set containing the origin (assumption 1 by Arrow and Hahn (1971)), the second, from $\mathbf{0} \in Y_{\mathbb{Z}}(n, 0, \ldots, 0)$ (assumption 1 once more), and the third, from $Y_{\mathbb{Z}}$'s superadditivity (their assumption 4). For each natural n, consider the rational profile $\boldsymbol{\chi}_n = (1, 1/n, \ldots, 1/n)$, so that, in light of the homogeneity property expressed in Arrow and Hahn (1971, assumption 3), what we have shown is that $\mathbf{y} \in (1/n)\,Y_{\mathbb{R}}(n, 1, \ldots, 1) = Y_{\mathbb{R}}(\boldsymbol{\chi}_n), \forall n \in \mathbb{N}$. By the continuity—more precisely, the upper semicontinuity—of $Y_{\mathbb{R}}$ prescribed in that same assumption, since $\boldsymbol{\chi}_n \to (1, 0, \ldots, 0)$, we must then also have $\mathbf{y} \in Y_{\mathbb{R}}(1, 0, \ldots, 0) = Y(\{1\})$.[21] Such upper semicontinuity is then employed to derive the homogeneity and superadditivity properties of the $Y_{\mathbb{R}}$ (in our notation) extension used in their version of the Core Equivalence Theorem.

Therefore, the announced generalization in the approach by Arrow and Hahn (1971) to the simplifying hypotheses of Debreu and Scarf (1963) of a constant returns and publicly accessible technology, novel and clever as it is, should be taken with a grain of salt. In fact, satisfaction of Arrow and Hahn's assumptions implies that the technology available to any nonempty coalition $S \neq \mathcal{I}$ must include even all plans which, regardless of the scale of production, can be produced by the complementary coalition $\mathcal{I} \setminus S$:

$$Y(S) \supseteq \sum_{i \in S} Y(\{i\}) \supseteq \sum_{i \in S} \mathrm{A}\,Y(\mathcal{I} \setminus \{i\}) \supseteq \sum_{i \in S} \mathrm{A}\,Y(\mathcal{I} \setminus S) = \mathrm{A}\,Y(\mathcal{I} \setminus S).$$

An immediate and notably restrictive conclusion with respect to the "asymptotic" accessibility of technology in Arrow and Hahn's model would be that $\mathrm{A}\,Y(S) = \mathrm{A}\,Y(T)$ for any

possible profiles, whilst requiring that the extended correspondence has certain properties (in particular, continuity and homogeneity). The space of profiles is not inherently defined, but the properties required depend amongst other things on this space; it should therefore be chosen with care."

[21] Compact-valuedness is obviously not assumed in their definition of upper semicontinuity.

two disjoint nonempty coalitions S and T:

$$\mathrm{A}\,Y\left(T\right) \subseteq \mathrm{A}\,Y\left(\mathcal{I} \setminus S\right) = \mathrm{A}\left(\mathrm{A}\,Y\left(\mathcal{I} \setminus S\right)\right) \subseteq \mathrm{A}\,Y\left(S\right),$$

and similarly for the opposite inclusion.[22] If only constant returns technologies were being considered, then this constraint on technology accessibility would take the even sharper form "$Y\left(S\right) = Y\left(T\right)$ for any two disjoint nonempty coalitions S and T," almost the same as in the analysis of Debreu and Scarf (1963).[23,24]

In the next section, we circumvent the need to assume the existence of a continuous technology correspondence by introducing an original technique for extending $Y_{\mathbb{Q}}$ in a constructive manner. Although lower semicontinuous per construction, it will typically—that is, unless the original economy presents enough productive uniformity as explained above—not be continuous. Coupled with the arguments in Arrow and Hahn (1971), this constructive, rather than axiomatic, approach to $Y_{\mathbb{R}}$ could also be used to yield a version of core equivalence for arbitrary sequences of type economies.

4 Core Equivalence Theorem

Let us now present proofs for each direction of the Core Equivalence Theorem. We start with the following key theorem.

Theorem 1 (First Welfare Theorem)**.** Given an economy $\mathcal{E} = \left(\mathcal{I}, \left(\mathbf{e}^i\right)_{i\in\mathcal{I}}, \left(\succsim^i\right)_{i\in\mathcal{I}}, Y\right)$, $W\left(\mathcal{E}\right) \subseteq C\left(\mathcal{E}\right)$.

Proof. Let $\mathbf{x} \in W\left(\mathcal{E}\right)$. First, note from E4 that $\mathbf{x}$ is $\mathcal{I}$-feasible. Now, suppose by contradiction that there exists a coalition S of $\mathcal{E}$ that blocks $\mathbf{x}$. That is, there exists an S-feasible allocation $\mathbf{x}'$ such that $\mathbf{x}'^i \succ^i \mathbf{x}^i$ for all $i \in S$.

Letting $\left(\mathbf{t}, \mathbf{p}, \mathbf{y}\right) \in \mathbb{R}^I_+ \times \mathbb{R}^L_{++} \times Y\left(\mathcal{I}\right)$ be such that $\left(\mathbf{x}, \mathbf{t}, \mathbf{p}, \mathbf{y}\right)$ is a Walras equilibrium, it is clear by E3 that, for all $i \in S$, $\mathbf{p} \cdot \mathbf{x}'^i > \mathbf{p} \cdot \mathbf{e}^i + t^i$ due to the strict monotonicity of

[22] We are here employing the facts that Y is monotonic and A is both monotonic and idempotent.

[23] The difference being that at least Arrow and Hahn (1971) would allow for the grand coalition $\mathcal{I}$ to be strictly more productive than all the other (equally productive) coalitions.

[24] A similar argument can be used to show that, if $Y(\mathcal{I})$ presents constant returns to scale, assuming the existence of a continuous extension of $Y_{\mathbb{Q}}$ to $\mathbb{R}^I_+$ would also imply that $Y(\mathcal{I}) \subseteq Y(\{i\})$ for all $i \in \mathcal{I}$.

preferences. By adding up these inequalities one thus obtains

$$\mathbf{p} \cdot \sum_{i \in S} \left(\mathbf{x}'^i - \mathbf{e}^i\right) > \sum_{i \in S} t^i.$$

Let $\mathbf{y}' = \sum_{i \in S} \left(\mathbf{x}'^i - \mathbf{e}^i\right)$, which belongs to $Y(S)$ due to S-feasibility of $\mathbf{x}'$. By definition of $\Pi_{\mathcal{E},\mathbf{p}}(S)$ and E2, we have $\mathbf{p} \cdot \mathbf{y}' \leq \Pi_{\mathcal{E},\mathbf{p}}(S) \leq \sum_{i \in S} t^i$. Therefore,

$$\sum_{i \in S} t^i \geq \mathbf{p} \cdot \mathbf{y}' = \mathbf{p} \cdot \sum_{i \in S} \left(\mathbf{x}'^i - \mathbf{e}^i\right) > \sum_{i \in S} t^i,$$

a contradiction. □

If an economy $\mathcal{E}$ satisfies TA1, so that the sequence of replica economies $(\mathcal{E}_r)_{r \in \mathbb{N}}$ adheres to the equal treatment lemma, we can establish that the sequence $(W(\mathcal{E}_r))_{r \in \mathbb{N}}$ also conforms to the equal treatment lemma by applying the above theorem to each replica. As a result, the sequence $(W_r(\mathcal{E}))_{r \in \mathbb{N}}$ of subsets of $\mathbb{R}^{LI}$ is well defined, and is clearly also a chain in the partial order of set inclusion. By taking the intersection over all natural rs, we conclude from Theorem 1 that

$$\bigcap_{r=1}^{\infty} W_r(\mathcal{E}) \subseteq \bigcap_{r=1}^{\infty} C_r(\mathcal{E}),$$

thereby confirming the First Welfare Theorem direction of core equivalence.

That $(W_r(\mathcal{E}))_{r \in \mathbb{N}}$ may in fact be a non-constant decreasing sequence is shown in Example 3 in Appendix A. As pointed out before, this is due to superadditivity not in the base economy, but in the very replication structure. Although absent from the literature, this potential non-constancy of the $(W_r(\mathcal{E}))_{r \in \mathbb{N}}$ sequence is stressed here because, as showcased in the aforementioned example, it is generally *not* the case that $W_1(\mathcal{E}) \subseteq \bigcap_{r=1}^{\infty} C_r(\mathcal{E})$.[25]

We now proceed to establish the converse direction of the Core Equivalence Theorem, that of Edgeworth's Conjecture. In the forthcoming proof, in contrast to Arrow and Hahn (1971), we abstain from the previously discussed assumption regarding boundedness of

[25] If the technology correspondences Y_r were assumed to be additive across levels of the replica economies—thus implying, for instance, $Y_{\mathbb{Z}}(3,2,5) = Y_{\mathbb{Z}}(1,1,1) + Y_{\mathbb{Z}}(1,1,1) + Y_{\mathbb{Z}}(1,0,1) + Y_{\mathbb{Z}}(0,0,1) + Y_{\mathbb{Z}}(0,0,1)$—, then, as shown in a previous version of this work, $W_r(\mathcal{E}) = W(\mathcal{E}), \forall r \in \mathbb{N}$, indeed. We thank Gil Riella and Charles Zheng for encouraging our focus on the present, more general approach to the core equivalence result.

nonconvexities and the requirement of existence of a continuous extension to $Y_{\mathbb{Q}}$. As discussed in the preceding section, the latter assumption is quite restrictive on the breadth of heterogeneity allowed in the original economy.

Moreover, as a minor divergence from Debreu and Scarf (1963) and Böhm (1974a), we do not mandate individual endowments to be strictly positive. Instead, we adopt the more general technical assumption TA2. Given that inaction is permissible in our context, it is also more accommodating than that of Xiong and Zheng (2007), which posits that $\sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$.

One final note on the upcoming proof: while we incorporate the clever approach from Arrow and Hahn (1971) in defining an auxiliary economy, whose individuals are envisaged as productive resources, our line of argumentation distinctly departs from theirs. More precisely, once the auxiliary economy is introduced, our reasoning more closely aligns with Debreu and Scarf (1963).

Theorem 2 (Edgeworth's Conjecture)**.** Given a base economy $\mathcal{E} = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y\right)$ and a sequence of replica economies $(\mathcal{E}_r)_{r\in\mathbb{N}} = \left(\mathcal{I}_r, (\mathbf{e}^i)_{(i,q)\in\mathcal{I}_r}, (\succsim^i)_{(i,q)\in\mathcal{I}_r}, Y_r\right)_{r\in\mathbb{N}}$ of $\mathcal{E}$ satisfying TA1, TA2 and TA3,

$$\bigcap_{r=1}^{\infty} C_r(\mathcal{E}) \subseteq \bigcap_{r=1}^{\infty} W_r(\mathcal{E}).$$

More specifically, there exists a price vector $\mathbf{p} \in \mathbb{R}^L_{++}$ and a profit distribution $(t^i)_{i\in\mathcal{I}}$ such that, given any $r \in \mathbb{N}$, $\left({}^r(\mathbf{x}^i)_{i\in\mathcal{I}}, {}^r(t^i)_{i\in\mathcal{I}}, \mathbf{p}, r\overline{\mathbf{y}}\right)$ is a Walrasian equilibrium in $\mathcal{E}_r$.

Proof. Let us further extend the domain of $Y_{\mathbb{Q}}$ to real-valued vectors. For a vector $\mathbf{s} \in \mathbb{R}^I_+$, we define

$$Y_{\mathbb{R}}(\mathbf{s}) = \overline{\bigcup_{\boldsymbol{\chi}\in\mathbb{Q}^I_+,\, \boldsymbol{\chi}\leq\mathbf{s}} Y_{\mathbb{Q}}(\boldsymbol{\chi})},$$

where the union is performed over all rational profiles less than or equal to $\mathbf{s}$. Given the monotonicity of $Y_{\mathbb{Q}}$, it is clear that $Y_{\mathbb{Q}}$ and $Y_{\mathbb{R}}$ coincide for rational vectors.[26]

[26]The reader might wonder why we take the closure in the definition of $Y_{\mathbb{R}}$. This is because the closedness of each $Y_{\mathbb{R}}(\mathbf{s})$ is necessary for the proofs of Claims 4 and 5, which concern the homogeneity and superadditivity of $Y_{\mathbb{R}}$. Assumption TA3 plays a crucial role here, as it guarantees that $Y_{\mathbb{Q}}$ is closed-valued, ensuring that $Y_{\mathbb{R}}$ and $Y_{\mathbb{Q}}$ indeed coincide for rational vectors.

Inspired by the insight of Arrow and Hahn (1971), let

$$Z = \left\{(\mathbf{y}, -\mathbf{s}) \in \mathbb{R}^L \times \mathbb{R}^I_- : \mathbf{y} \in Y_{\mathbb{R}}(\mathbf{s})\right\}.$$

By Claim 6 in Appendix B, the set Z is a convex cone.[27] Define the following economy:

$$\mathcal{E}' = \left(\mathcal{I}, \left(\mathbf{f}^i\right)_{i\in\mathcal{I}}, \left(\trianglerighteq^i\right)_{i\in\mathcal{I}}, \overline{Y}\right),$$

where $\overline{Y} : 2^{\mathcal{I}} \rightrightarrows \mathbb{R}^{L+I}$ is defined by $\overline{Y}(\varnothing) = \{\mathbf{0}\}$ and $\overline{Y}(S) = Z$ for all $S \in 2^{\mathcal{I}} \setminus \{\varnothing\}$, that is, each nonempty coalition has access to the same production set Z (the economy $\mathcal{E}'$ has publicly accessible technology, as in Debreu and Scarf, 1963), the endowment is $\mathbf{f}^i = (\mathbf{e}^i, \mathbf{u}^i)$ ($\mathbf{u}^i$ is the i-th vector of the canonical basis of $\mathbb{R}^I$) and $\trianglerighteq^i$ is a preference relation on $\mathbb{R}^{L+I}_+$ that considers $\succsim^i$ for the initial L coordinates and disregards the remaining.[28]

For each $r \in \mathbb{N}$, define $\mathcal{E}'_r := \left(\mathcal{I}_r, \left(\mathbf{f}^i\right)_{(i,q)\in\mathcal{I}_r}, \left(\trianglerighteq^i\right)_{(i,q)\in\mathcal{I}_r}, \overline{Y}_r\right)$, where $\overline{Y}_r : 2^{\mathcal{I}_r} \rightrightarrows \mathbb{R}^{L+I}$ is given by $\overline{Y}_r(\varnothing) = \{\mathbf{0}\}$ and $\overline{Y}_r(S) = Z$ for all $S \in 2^{\mathcal{I}_r} \setminus \{\varnothing\}$. It is straightforward to verify that $(\mathcal{E}'_r)_{r\in\mathbb{N}}$ defines a sequence of replica economies of $\mathcal{E}'$.

Let $\left(\mathbf{x}^i\right)_{i\in\mathcal{I}} \in \bigcap_{r=1}^{\infty} C_r(\mathcal{E})$. As outlined in Claim 7 in Appendix B, for every $r \in \mathbb{N}$, if we define $\mathbf{x}^{(i,q)} = \mathbf{x}^i, \forall q \in \{1, \ldots, r\}$ $\left(\mathbf{x}^{(i,q)}, \mathbf{0}\right)_{(i,q)\in\mathcal{I}_r} \in C(\mathcal{E}'_r)$. Define Γ as the convex hull of the union of the following I sets:

$$\Gamma^i = \left\{\mathbf{g} \in \mathbb{R}^{L+I} : \mathbf{g} + \mathbf{f}^i \vartriangleright^i \left(\mathbf{x}^i, \mathbf{0}\right)\right\}.$$

Assume towards a contradiction that there exists $\boldsymbol{\gamma} \in \Gamma \cap Z$. Since $\boldsymbol{\gamma} \in \Gamma$, we can write $\boldsymbol{\gamma} = \sum_{i\in\mathcal{I}} \alpha^i \mathbf{z}^i$, where, for each $i \in \mathcal{I}$, $\alpha^i \geq 0$, and $\mathbf{z}^i + \mathbf{f}^i \vartriangleright^i (\mathbf{x}^i, \mathbf{0})$ for each $i \in \mathcal{I}$ such that $\alpha^i > 0$. Let $S = \{i \in \mathcal{I} : \alpha^i > 0\}$.

For each $i \in S$ and $n \geq 0$, define

$$\mathbf{z}^{i,n} = \frac{n\alpha^i}{\lceil n\alpha^i \rceil}\mathbf{z}^i.$$

[27] If, instead of defining Z using the constructed extension $Y_{\mathbb{R}}$, we adopted the approach of Arrow and Hahn (1971), which uses a Y extension assumed to be continuous (see Section 3), the set Z would be closed. As a consequence, TA3 could be regarded as a standard assumption defining an economy, and $\mathcal{E}'$ would still qualify as one.

[28] Because $\mathcal{I}$ is finite, each individual of the original economy can be associated to an integer between 1 and I.

Observe that, for each $i \in S$, $(\mathbf{z}^{i,n} + \mathbf{f}^i)_{n\in\mathbb{N}}$ converges to $\mathbf{z}^i + \mathbf{f}^i$ and, by lower semicontinuity, there is a n_0 large enough such that

$$\mathbf{z}^{i,n_0} + \mathbf{f}^i \rhd^i \left(\mathbf{x}^i, \mathbf{0}\right) \text{ for all } i \in S.$$

We now argue that any coalition J in $\mathcal{E}'_{n_0}$ that contains $\lceil n_0\alpha^i \rceil$ members of each name $i \in S$ blocks $(\mathbf{x}^i, \mathbf{0})_{(i,q)\in\mathcal{I}_r}$ by allocating to each member of name i the bundle $\mathbf{z}^{i,n_0} + \mathbf{f}^i$. Since we already know that each member of J is already better off with $\mathbf{z}^{i,n_0} + \mathbf{f}^i$ than with $(\mathbf{x}^i, \mathbf{0})$, it is enough to check for the J-feasibility of this allocation:

$$\sum_{i\in\mathcal{I}} \left(\lceil n_0\alpha^i \rceil \left(\mathbf{z}^{i,n_0} + \mathbf{f}^i\right) - \lceil n_0\alpha^i \rceil \mathbf{f}^i\right) = \sum_{i\in\mathcal{I}} \lceil n_0\alpha^i \rceil \mathbf{z}^{i,n_0} = \sum_{i\in\mathcal{I}} n_0\alpha^i\mathbf{z}^i = n_0 \sum_{i\in\mathcal{I}} \alpha^i\mathbf{z}^i = n_0\gamma \in Z.$$

Therefore, $(\mathbf{x}^i, \mathbf{0})_{(i,q)\in\mathcal{I}_r}$ is indeed blocked by J, a contradiction, since $(\mathbf{x}^i, \mathbf{0})_{(i,q)\in\mathcal{I}_r} \in C\left(\mathcal{E}'_r\right)$, by Claim 7.

Since Γ and Z are convex, nonempty and Z is a cone, there is a hyperplane that separates them and goes through the origin (see, for example, theorems 11.3 and 11.7 of Rockafellar, 1970). That is, there exists a non-null vector $(\mathbf{p}, \mathbf{t}) \in \mathbb{R}^L \times \mathbb{R}^I$ satisfying

$$(\mathbf{p}, \mathbf{t}) \cdot \mathbf{z} \leq 0 \text{ for all } \mathbf{z} \in Z \tag{1}$$

and

$$(\mathbf{p}, \mathbf{t}) \cdot \mathbf{g} \geq 0 \text{ for all } \mathbf{g} \in \Gamma. \tag{2}$$

Let $\overline{\mathbf{y}} = \sum_{i\in\mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i)$. Given any $r \in \mathbb{N}$, it will be shown that $\left({}^r\left(\mathbf{x}^i\right)_{i\in\mathcal{I}}, {}^r\left(t^i\right)_{i\in\mathcal{I}}, \mathbf{p}, r\overline{\mathbf{y}}\right)$ is a Walrasian equilibrium in $\mathcal{E}_r$, thus ending the proof. So we now verify each equilibrium condition.

(E.4) Because $\left(\mathbf{x}^i\right)_{i\in\mathcal{I}}$ is feasible in $\mathcal{E}$, we may already conclude that $\overline{\mathbf{y}} \in Y(\mathcal{I})$. Then, by homogeneity of $Y_{\mathbb{Z}}$, $r\overline{\mathbf{y}}$ is feasible in $\mathcal{E}_r$.

(E.2) Let S be an arbitrary coalition of $\mathcal{E}_r$ and let $\mathbf{y}^S \in Y_{\mathbb{Z}}(s)$, where $\kappa(S) = \mathbf{s}$. Then, since $Y_{\mathbb{Z}}$ and $Y_{\mathbb{R}}$ coincide over integer-valued vectors, $\left(\mathbf{y}^S, -\mathbf{s}\right) \in Z$, that is, $(\mathbf{p}, t) \cdot \left(\mathbf{y}^S, -\mathbf{s}\right) \leq 0$, meaning $\sum\limits_{i\in S} t^i \geq \mathbf{p} \cdot \mathbf{y}^S$. Given that $\mathbf{y}^S$ was chosen arbitrarily in $Y_{\mathbb{Z}}(\mathbf{s})$, it follows that $\sum\limits_{i\in S} t^i \geq \Pi_{\mathcal{E}_r,\mathbf{p}}(S)$. Specifically, for each $i \in \mathcal{I}$, by setting S as the coalition containing only the individual i and realizing that $Y_{\mathbb{Z}}$ is inaction allowing, one obtains $t^i \geq 0$. Additionally, by taking $S = \mathcal{I}$, $\sum\limits_{i\in S} t^i \geq \Pi_{\mathcal{E},\mathbf{p}}(\mathcal{I}) \geq \mathbf{p} \cdot \overline{\mathbf{y}} = \mathbf{p} \cdot \sum_{i\in\mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i)$.

(E.3) First note that, for each $i \in \mathcal{I}$ (and not only in the aggregate), $\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + t^i$. In fact, strong monotonicity of preferences yields, for all $n \in \mathbb{N}$, $(\mathbf{x}^i + (1/n)\mathbf{u}^1, \mathbf{0}) - \mathbf{f}^i \in \Gamma^i \subseteq \Gamma$, where $\mathbf{u}^1$ is the first vector of the canonical basis of $\mathbb{R}^L$. Therefore, $(\mathbf{p}, t) \cdot ((\mathbf{x}^i + (1/n)\mathbf{u}^1, \mathbf{0}) - \mathbf{f}^i) \geq 0$ by 2, that is, $\mathbf{p} \cdot (\mathbf{x}^i + (1/n)\mathbf{u}^1) \geq \mathbf{p} \cdot \mathbf{e}^i + t^i$. Continuity of the inner product then gives $\mathbf{p} \cdot (\mathbf{x}^i - \mathbf{e}^i) \geq t^i$. However, if for any $i \in \mathcal{I}$, this inequality were strict, we would end up with $\mathbf{p} \cdot \sum_{i \in \mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i) > \sum_{i \in \mathcal{I}} t^i$, in contradiction with what has already been established with respect to E.2. Thus, actually $\mathbf{p} \cdot (\mathbf{x}^i - \mathbf{e}^i) = t^i$ for all $i \in \mathcal{I}$.

We now argue that, given $i \in \mathcal{I}$, whenever $\mathbf{x}'^i \succ^i \mathbf{x}^i$, $\mathbf{p} \cdot \mathbf{x}'^i > \mathbf{p} \cdot \mathbf{e}^i + t^i$ holds. To support this argument, we refer to Claim 8 in Appendix B, which states that $\mathbf{p}$ is strictly positive, that is, $\mathbf{p} \gg \mathbf{0}$.

Assume, by contradiction, that $\mathbf{x}'^i \succ^i \mathbf{x}^i$ and $\mathbf{p} \cdot \mathbf{x}'^i = \mathbf{p} \cdot \mathbf{e}^i + t^i$. Leveraging the property of lower semicontinuity, we can identify a vector $\mathbf{x}''^i$ such that $\mathbf{x}''^i \succ^i \mathbf{x}^i$, $\mathbf{x}''^i \leq \mathbf{x}'^i$ and $\mathbf{x}''^i \neq \mathbf{x}'^i$. Thus, since $\mathbf{p} \gg \mathbf{0}$, we set $\mathbf{p} \cdot \mathbf{x}''^i < \mathbf{p} \cdot \mathbf{e}^i + t^i$, a contradiction.

Finally, as pointed out in Section 2, E.1 follows immediately from E.3 and E.4 due to local nonsatiation. □

By combining the results from the previous theorems, one obtains, for an economy $\mathcal{E}$ and a sequence of replica economies $(\mathcal{E}_r)_{r \in \mathbb{N}}$ satisfying TA1, TA2 and TA3,

$$\bigcap_{r=1}^{\infty} C_r(\mathcal{E}) = \bigcap_{r=1}^{\infty} W_r(\mathcal{E}),$$

the Core Equivalence result.

5 Fuzzy extension

The reader may wonder what would occur if individuals could have fractional participation rates in firms, as in Aubin (1979, ch. 10). The issue of core equivalence with this divisibility feature taken into account in the context of additive—and, in particular, private ownership—economies is addressed by Aubin (1979, 1981) (for the infinitely many commodities case, see Florenzano, 1990). Here, we show that the same method used in the previous section lends itself to providing core equivalence in the more general context of superadditive fuzzy coalition production economies.

Definition 6. Let there be given a finite nonempty set of individuals $\mathcal{I}$ (of cardinality I as before), each $i \in \mathcal{I}$ with an endowment $\mathbf{e}^i \in \mathbb{R}_+^L$ and a rational preference relation $\succsim^i$, and a technology correspondence $Y_f : [0,1]^I \rightrightarrows \mathbb{R}^L$ mapping each fuzzy coalition into its production possibility set. We shall refer to $\mathcal{E}^f = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y_f\right)$ as a *fuzzy coalition production economy* (or, for short, a *fuzzy economy*) if the following assumptions hold:

AF1. For all $i \in \mathcal{I}$, $\succsim^i$ is lower semicontinuous, strictly monotone (i.e., $\mathbf{x}^i \geq \mathbf{x}'^i$ implies $\mathbf{x}^i \succsim^i \mathbf{x}'^i$ and $\mathbf{x}^i \gg \mathbf{x}'^i$ implies $\mathbf{x}^i \succ^i \mathbf{x}'^i$), and convex, over the consumption set $\mathbb{R}_+^L$;

AF2. For all $\mathbf{s} \in [0,1]^I$, $Y_f(\mathbf{s})$ is convex and contains $\mathbf{0}$ (and only $\mathbf{0}$, if $s = \mathbf{0}$);

AF3. For all $\mathbf{s}, \mathbf{s}' \in [0,1]^I$ such that $\mathbf{s} + \mathbf{s}' \in [0,1]^I$, $Y_f(\mathbf{s} + \mathbf{s}') \supseteq Y_f(\mathbf{s}) + Y_f(\mathbf{s}')$;

AF4. For all $\mathbf{s} \in [0,1]^I$ and $\alpha \in \mathbb{R}_+$ such that $\alpha\mathbf{s} \in [0,1]^I$, $Y_f(\alpha\mathbf{s}) = \alpha Y_f(\mathbf{s})$.

Assumptions AF1–AF3 mirror A1–A3 from Section 2. As before, since each production possibility set contains $\mathbf{0}$, the superadditivity of Y_f also yields its monotonicity. Assumption AF4 parallels the homogeneity property presented earlier.

As in the non-fuzzy case, these hypotheses do not imply additivity of Y_f. For instance, consider an economy with two individuals and two goods. Let $Y := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0 \text{ and } x_2 \leq \sqrt{-x_1}\}$. By defining, for each fuzzy coalition $(a,b) \in [0,1]^2$, $Y_f(a,b) = \min\{a,b\}Y$, we obtain a Y_f that is superadditive and homogeneous, but not additive: $(-4,2)$ belongs to $Y_f(1,1)$ $(= Y)$, but not to $Y_f(1,0) + Y_f(0,1)$ $(= \{(0,0)\})$.

As in Section 2, we now introduce a couple of technical assumptions, which will be required in the proof of Theorem 4:

TAF1. For all $i \in \mathcal{I}$, $\succsim^i$ is strictly convex over the consumption set $\mathbb{R}_+^L$;

TAF2. There exists a $\widetilde{\mathbf{y}} \in Y_f(\mathbf{1})$ such that $\sum_{i\in\mathcal{I}} \mathbf{e}^i + \widetilde{\mathbf{y}} \gg \mathbf{0}$.[29]

Assumption TAF1 mirrors TA1, while TAF2 is a direct extension of TA2 to the fuzzy context. However, contrasting from the replicas case, closed valuedness of the production correspondence (A3) is not necessary for the validity of core equivalence.

[29] Here, $\mathbf{1}$ stands for $(1,\ldots,1) \in \mathbb{R}^I$.

Given a non-null coalition $\mathbf{s}$, define $car(\mathbf{s}) = \{i \in \mathcal{I} : s^i > 0\}$, the set of individuals that participate in $\mathbf{s}$. An allocation $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{I}}$ is $\mathbf{s}$-*feasible* if $\sum_{i\in\mathcal{I}} s^i (\mathbf{x}^i - \mathbf{e}^i) \in Y_f(\mathbf{s})$ (and just *feasible* if it is $\mathbf{1}$-feasible, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^I$). A feasible allocation $(\mathbf{x}^i)_{i\in\mathcal{I}}$ is *blocked* by $\mathbf{s}$ if there exists an $\mathbf{s}$-feasible allocation $\mathbf{x}' = (\mathbf{x}'^i)_{i\in\mathcal{I}}$ such that $\mathbf{x}'^i \succ^i \mathbf{x}^i$ for all $i \in car(\mathbf{s})$.

Definition 7. The *core* of $\mathcal{E}^f$ is the set $C\left(\mathcal{E}^f\right)$ of all feasible allocations of $\mathcal{E}^f$ that are not blocked by any coalition of that economy.

Let $\mathbf{p} \in \mathbb{R}^L_{++}$. Similar to the potential profit function defined in Section 2, we define the *fuzzy potential profit function* $\Pi_{\mathcal{E}^f,\mathbf{p}} : [0,1]^I \to [0,\infty]$ by $\Pi_{\mathcal{E}^f,\mathbf{p}}(\mathbf{s}) = \sup_{\boldsymbol{v}\in Y_f(\mathbf{s})} \mathbf{p}\cdot\boldsymbol{v}$.

The definition of Walras equilibrium presented below mirrors Definition 3, with the main difference being that the profit distribution $(t^i)_{i\in\mathcal{I}}$ must now lie in the core of the fuzzy game with side payments (Aubin, 1981) defined by $\left(\mathcal{I}, \Pi_{\mathcal{E}^f,\mathbf{p}}\right)$.

Definition 8. Given a fuzzy economy $\mathcal{E}^f = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y_f\right)$, a *Walras (or competitive) equilibrium* of $\mathcal{E}^f$ is a tuple $(\mathbf{x}, \mathbf{t}, \mathbf{p}, \mathbf{y}) \in \mathbb{R}^{LI}_+ \times \mathbb{R}^I_+ \times \mathbb{R}^L_{++} \times Y_f(\mathbf{1})$ such that:

EF1. $\sum\limits_{i\in\mathcal{I}} t^i \le \mathbf{p}\cdot\mathbf{y}$;

EF2. For all $\mathbf{s} = (s^i)_{i\in\mathcal{I}} \in [0,1]^I$, $\sum\limits_{i\in\mathcal{I}} s^i t^i \ge \Pi_{\mathcal{E}^f,\mathbf{p}}(\mathbf{s})$;

EF3. For all $i \in \mathcal{I}$, $\mathbf{x}^i$ is a maximal element of $\left\{\mathbf{c}^i \in \mathbb{R}^L_+ : \mathbf{p}\cdot\mathbf{c}^i \le \mathbf{p}\cdot\mathbf{e}^i + t^i\right\}$ with respect to $\succsim^i$ and

EF4. $\sum\limits_{i\in\mathcal{I}} \mathbf{x}^i = \sum\limits_{i\in\mathcal{I}} \mathbf{e}^i + \mathbf{y}$.

We now present the initial direction of the fuzzy core equivalence result.

Theorem 3 (First Welfare Theorem)**.** Given a fuzzy economy $\mathcal{E}^f = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y_f\right)$, $W\left(\mathcal{E}^f\right) \subseteq C\left(\mathcal{E}^f\right)$.

Proof. Let $\mathbf{x} \in W\left(\mathcal{E}^f\right)$. First, note from EF4 that $\mathbf{x}$ is $\mathbf{1}$-feasible. Now, suppose by contradiction that there exists a non-null coalition $\mathbf{s} = (s^i)_{i\in\mathcal{I}}$ of $\mathcal{E}^f$ that blocks $\mathbf{x}$. That is, there exists an $\mathbf{s}$-feasible allocation $\mathbf{x}'$ such that $\mathbf{x}'^i \succ^i \mathbf{x}^i$ for all $i \in car(\mathbf{s})$.

Letting $(\mathbf{t}, \mathbf{p}, \mathbf{y}) \in \mathbb{R}^I_+ \times \mathbb{R}^L_{++} \times Y_f(\mathbf{1})$ be such that $(\mathbf{x}, \mathbf{t}, \mathbf{p}, \mathbf{y})$ is a Walras equilibrium, it is clear by EF3 that, for all $i \in car(\mathbf{s})$, $\mathbf{p} \cdot \mathbf{x}'^i > \mathbf{p} \cdot \mathbf{e}^i + t^i$ due to the strict monotonicity of preferences.

Define $\mathbf{y}' = \sum_{i\in\mathcal{I}} s^i (\mathbf{x}'^i - \mathbf{e}^i)$, which belongs to $Y_f(\mathbf{s})$ due to the $\mathbf{s}$-feasibility of $\mathbf{x}'$. By the definition of $\Pi_{\mathcal{E}^f,\mathbf{p}}(\mathbf{s})$ and EF2, we have $\mathbf{p} \cdot \mathbf{y}' \leq \Pi_{\mathcal{E}^f,\mathbf{p}}(\mathbf{s}) \leq \sum_{i\in\mathcal{I}} s^i t^i$. Therefore,

$$\sum_{i\in\mathcal{I}} s^i t^i \geq \mathbf{p} \cdot \mathbf{y}' = \mathbf{p} \cdot \sum_{i\in\mathcal{I}} s^i \left(\mathbf{x}'^i - \mathbf{e}^i\right) = \sum_{i\in\mathcal{I}} s^i \mathbf{p} \cdot \left(\mathbf{x}'^i - \mathbf{e}^i\right) > \sum_{i\in\mathcal{I}} s^i t^i,$$

a contradiction. □

The converse is established in the following theorem. The proof follows a similar structure to that of theorem 2, with modifications to accommodate the fuzzy environment. The main difference lies in the definition of the extension $Y_\mathbb{R}$, which now extends Y_f, defined on the domain $[0,1]^I$.

Theorem 4 (Edgeworth's Conjecture)**.** Given a fuzzy economy $\mathcal{E}^f = \left(\mathcal{I}, (\mathbf{e}^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y_f\right)$, $C\left(\mathcal{E}^f\right) \subseteq W\left(\mathcal{E}^f\right)$.

Proof. We start by defining an auxiliary function $Y_\mathbb{R}$. For each $\mathbf{s} \in \mathbb{R}^I_+$, let

$$Y_\mathbb{R}(s) = \begin{cases} \{\mathbf{0}\}, & \text{if } \mathbf{s} = \mathbf{0}; \\ \|\mathbf{s}\|_1 Y_f\left(\dfrac{1}{\|\mathbf{s}\|_1}\mathbf{s}\right), & \text{otherwise,} \end{cases}$$

where $\|\mathbf{s}\|_1 = \sum_{i\in\mathcal{I}} s^i$.

In the same manner as the proof of theorem 2, define the set

$$Z = \left\{(\mathbf{y}, -\mathbf{s}) \in \mathbb{R}^L \times \mathbb{R}^I_- : \mathbf{y} \in Y_\mathbb{R}(\mathbf{s})\right\}.$$

Appendix B provides proofs that $Y_\mathbb{R}$ is homogeneous (Claim 9), an extension of Y_f (Claim 10), and superadditive (Claim 11). These properties allow us to demonstrate that Z is a convex cone (Claim 12). As in the proof of theorem 2, and making use of the same notation therein established for $\mathbf{f}^i$ and $\trianglerighteq^i$, consider the fuzzy economy:

$$\mathcal{E}^{f'} = \left(\mathcal{I}, \left(\mathbf{f}^i\right)_{i\in\mathcal{I}}, \left(\trianglerighteq^i\right)_{i\in\mathcal{I}}, \overline{Y}\right),$$

where $\overline{Y} : [0,1]^I \rightrightarrows \mathbb{R}^{L+I}$ assigns each fuzzy coalition to the set Z.

Let $(\mathbf{x}^i)_{i\in\mathcal{I}} \in C\left(\mathcal{E}^f\right)$. By Claim 13 in Appendix B, we know that $(\mathbf{x}^i, \mathbf{0})_{i\in\mathcal{I}} \in C\left(\mathcal{E}^{f'}\right)$. Define Γ as the convex hull of the union of the following I sets:

$$\Gamma^i = \left\{\mathbf{g} \in \mathbb{R}^{L+I} : \mathbf{g} + \mathbf{f}^i \rhd^i \left(\mathbf{x}^i, \mathbf{0}\right)\right\}.$$

Assume towards a contradiction that there exists $\boldsymbol{\gamma} \in \Gamma \cap Z$. Since $\boldsymbol{\gamma} \in \Gamma$, we can write $\boldsymbol{\gamma} = \sum_{i\in\mathcal{I}} \alpha^i \mathbf{z}^i$, where $\sum_{i\in\mathcal{I}} \alpha^i = 1$, for each $i \in \mathcal{I}$, $\alpha^i \geq 0$, and $\mathbf{z}^i + \mathbf{f}^i \rhd^i (\mathbf{x}^i, \mathbf{0})$ for each $i \in \mathcal{I}$ such that $\alpha^i > 0$. Let $\mathbf{s} = (\alpha^i)_{i\in\mathcal{I}}$.

We now argue that the fuzzy coalition $\mathbf{s}$ blocks $(\mathbf{x}^i, \mathbf{0})_{i\in\mathcal{I}}$ by allocating to each member i the bundle $\mathbf{z}^i + \mathbf{f}^i$. Since we already know that each member of $car(\mathbf{s})$ is already better off with $(\mathbf{z}^i + \mathbf{f}^i)_{i\in\mathcal{I}}$ allocation than with $(\mathbf{x}^i, \mathbf{0})$, it is enough to check for the $\mathbf{s}$-feasibility of this allocation:

$$\sum_{i\in\mathcal{I}} \alpha^i \left(\left(\mathbf{z}^i + \mathbf{f}^i\right) - \mathbf{f}^i\right) = \sum_{i\in\mathcal{I}} \alpha^i \mathbf{z}^i = \boldsymbol{\gamma} \in Z.$$

Therefore, $(\mathbf{x}^i, \mathbf{0})_{i\in\mathcal{I}}$ is indeed blocked by $\mathbf{s}$, a contradiction, since $(\mathbf{x}^i, \mathbf{0})_{i\in\mathcal{I}}$ belongs to the core of $\mathcal{E}^{f'}$ by Claim 13. As in the proof of theorem 2, we can conclude that there exists a nonnull vector $(\mathbf{p}, \mathbf{t}) \in \mathbb{R}^L \times \mathbb{R}^I$ satisfying

$$(\mathbf{p}, \mathbf{t}) \cdot \mathbf{z} \leq 0 \text{ for all } \mathbf{z} \in Z \tag{3}$$

and

$$(\mathbf{p}, \mathbf{t}) \cdot \mathbf{g} \geq 0 \text{ for all } \mathbf{g} \in \Gamma. \tag{4}$$

Let $\overline{\mathbf{y}} = \sum_{i\in\mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i)$. We will show that $\left((\mathbf{x}^i)_{i\in\mathcal{I}}, (t^i)_{i\in\mathcal{I}}, \mathbf{p}, \overline{\mathbf{y}}\right)$ is a Walrasian equilibrium in $\mathcal{E}^f$, thereby concluding the proof. We proceed to verify each equilibrium condition. Since $(\mathbf{x}^i)_{i\in\mathcal{I}}$ is feasible, we can already conclude that $\overline{\mathbf{y}} \in Y_f(\mathbf{1})$.

(EF4) This is warranted by the very definition of $\overline{\mathbf{y}}$.

(EF2) Let $\mathbf{s}$ be an arbitrary coalition of $\mathcal{E}^f$ and let $\mathbf{y}^{\mathbf{s}} \in Y_f(\mathbf{s})$. Then, by Claim 10, $(\mathbf{y}^{\mathbf{s}}, -\mathbf{s}) \in Z$, that is, $(\mathbf{p}, \mathbf{t}) \cdot (\mathbf{y}^{\mathbf{s}}, -\mathbf{s}) \leq 0$, which, in turn, gives $\sum\limits_{i\in car(\mathbf{s})} s^i t^i \geq \mathbf{p} \cdot \mathbf{y}^{\mathbf{s}}$. Since $\mathbf{y}^{\mathbf{s}}$ was chosen arbitrarily in $Y_f(\mathbf{s})$, we have $\sum\limits_{i\in car(\mathbf{s})} s^i t^i \geq \Pi_{\mathcal{E}^f, \mathbf{p}}(\mathbf{s})$. In particular, for each $i \in \mathcal{I}$, taking $\mathbf{s} = \mathbf{u}^i$ and realizing that Y_f is inaction allowing, one obtains $t^i \geq 0$. Yet another conclusion, this time plugging $\mathbf{s} = \mathbf{1}$, is $\sum\limits_{i\in\mathcal{I}} t^i \geq \Pi_{\mathcal{E}^f, \mathbf{p}}(\mathcal{I}) \geq \mathbf{p} \cdot \overline{\mathbf{y}} = \mathbf{p} \cdot \sum_{i\in\mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i)$.

(EF3) Follow the proof of this same condition in theorem 2.

As noted in Section 2, EF1 follows directly from EF3 and EF4 due to local nonsatiation, concluding the argument. □

6 Conclusion

We have presented a detailed investigation of the Core Equivalence Theorem in coalition production economies with decreasing returns to scale and superadditivity. We have shown that joint adaptation of Arrow and Hahn's (1971) idea of treating individuals as productive resources and incorporation of Debreu and Scarf's (1963) separation argument eliminates the need for certain restrictive assumptions regarding the production correspondence found in previous studies, including those by Arrow and Hahn (1971), Champsaur (1974), and Böhm (1974a).

Furthermore, we extend this proof to fuzzy coalition production economies, confirming that core equivalence remains applicable even in small economies, provided that individuals are allowed to participate fractionally in productive endeavors.

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A Appendix

Example 1. Consider the coalition production economy $\mathcal{E} = (\mathcal{I}, (e^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y)$, where $\mathcal{I} = \{A, B, C\}$ is the set of individuals, each with an endowment of $(1/3, 0)$ and identical preferences represented by the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ given by $u(x_1, x_2) = x_1 + 10\sqrt{x_2}$. The technology correspondence Y is defined as follows:

$$Y(S) = \begin{cases} \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0 \text{ and } x_2 \leq \sqrt{-x_1}\}, & \text{if } S = \mathcal{I}; \\ \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0 \text{ and } x_2 \leq (1/2)\sqrt{-x_1}\}, & \text{if } S = \{A, B\}; \\ \{\mathbf{0}\}, & \text{otherwise.} \end{cases}$$

If condition E2 is redefined to allow S to be either a singleton or the grand coalition, then the vector $\mathbf{t}$ in our definition of competitive equilibrium would only be required to be an imputation of the game $(\mathcal{I}, \Pi_{\mathcal{E},\mathbf{p}})$, and not necessarily a core member. Take $\mathbf{t} = (t^A, t^B, t^C) = (0, 0, 1)$, $\mathbf{p} = (1, 2)$, $\mathbf{y} = (-1, 1)$ and $\mathbf{x} = (\mathbf{x}^A, \mathbf{x}^B, \mathbf{x}^C) = ((0, 1/6), (0, 1/6), (0, 4/6))$. It can be shown that $(\mathbf{x}, \mathbf{t}, \mathbf{p}, \mathbf{y})$ would be a competitive equilibrium of $\mathcal{E}$. However, this allocation is blocked by coalition $\{A, B\}$, which is able to produce the bundle $(0, 1/2\sqrt{6})$ for each of its members.

Furthermore, by employing the definitions introduced in Section 2—specifically those of replica economies, the $Y_{\mathbb{Z}}$ correspondence, and the repetition notation—it is possible to see that even the limiting version of the First Welfare Theorem would fail. To illustrate this point, if we define $Y_{\mathbb{Z}}$ by using inter-replica additivity—for which, as an example, $Y_{\mathbb{Z}}(3, 7, 2) = 2Y_{\mathbb{Z}}(1, 1, 1) + Y_{\mathbb{Z}}(1, 1, 0) + 4Y_{\mathbb{Z}}(0, 1, 0)$—a sequence of correspondences $(Y_r)_{r\in\mathbb{N}}$ can be recovered and used to justify $(\mathcal{E}_r)_{r\in\mathbb{N}}$ as a sequence of replica economies of $\mathcal{E}$, for which all of the assumptions A.1–A.6 and TA.1–TA.3 hold. For every $r \in \mathbb{N}$, the profit distribution given by ${}^r\mathbf{t}$ is an imputation of the game $(\mathcal{I}_r, \Pi_{\mathcal{E}_r,\mathbf{p}})$, since, for all $(i, q) \in \mathcal{I}_r$, ${}^r t^{(i,q)} = t^i \geq \Pi_{\mathcal{E},\mathbf{p}}(\{i\}) = \Pi_{\mathcal{E}_r,\mathbf{p}}(\{(i, q)\})$ and

$$\sum_{(i,q)\in\mathcal{I}_r} {}^r t^{(i,q)} = r \sum_{i\in\mathcal{I}} t^i = r(\mathbf{p} \cdot \mathbf{y}) = \mathbf{p} \cdot r\mathbf{y} = \Pi_{\mathcal{E}_r,\mathbf{p}}(\mathcal{I}_r),$$

where the last equality comes from the homogeneity of $Y_{\mathbb{Z}}$ and the fact that $\mathbf{p}\cdot\mathbf{y} = \Pi_{\mathcal{E},\mathbf{p}}(\mathcal{I})$. By noting that the other competitive equilibrium conditions are consequently satisfied, we can conclude that $({}^r\mathbf{x}, {}^r\mathbf{t}, \mathbf{p}, r\mathbf{y})$ would be a competitive equilibrium for every replica $\mathcal{E}_r$. However, $\mathbf{x}$ does not belong to $\bigcap_{r=1}^{\infty} C_r(\mathcal{E})$, since it is not in $C(\mathcal{E})$.

Example 2. Consider the coalition production economy $\mathcal{E} = (\mathcal{I}, (e^i)_{i\in\mathcal{I}}, (\succsim^i)_{i\in\mathcal{I}}, Y)$, where $\mathcal{I} = \{A, B\}$, each individual is endowed with $(2, 0)$ and shares identical preferences, represented by the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$, defined as $u(x_1, x_2) = x_1 + 10\sqrt{x_2}$. The technology correspondence $Y : 2^{\mathcal{I}} \rightrightarrows \mathbb{R}^2$ is defined as follows:

$$Y(S) = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0,\ x_2 \leq -x_1 \text{ and } x_2 \leq |S|^2\right\},$$

if $S \neq \varnothing$ and $Y(S) = \{\mathbf{0}\}$ otherwise.

It is straightforward to verify that assumptions A1, A2 and A3 are satisfied. Let $\mathbf{t} = (t^A, t^B) = (3, 1)$, $\mathbf{p} = (1, 2)$, $\mathbf{y} = (-4, 4)$ and $\mathbf{x} = (\mathbf{x}^A, \mathbf{x}^B) = ((0, 5/2), (0, 3/2))$. We will show that $(\mathbf{x}, \mathbf{t}, \mathbf{p}, \mathbf{y})$ is a competitive equilibrium of $\mathcal{E}$.

To verify E2, observe that, because the price of good 2 is higher than that of good 1, the firm $\{A, B\}$ maximizes its profit by producing the largest feasible quantity of good 2, namely $|\{A, B\}|^2 = 4$ units, and thus chooses the production plan $\mathbf{y}$. As the maximum profit an individual can obtain alone is 1, E2 is confirmed.

To check E3, note that each individual $i \in \{A, B\}$ maximizes the utility function $u(x_1, x_2) = x_1 + 10\sqrt{x_2}$ subject to the budget constraint $x_1 + 2x_2 \leq 2 + t^i$ and $x_1, x_2 \geq 0$. Reformulating the problem in terms of x_2 alone, the objective function is increasing in x_2 as long as $x_2 \leq 25/4$. Since neither individual can afford this quantity, both will allocate their entire budget to good 2, resulting in the allocation $\mathbf{x}$.

Condition E4 holds trivially, since $\mathbf{x}^A + \mathbf{x}^B - \mathbf{e}^A - \mathbf{e}^B = (-4, 4) = \mathbf{y}$. Finally, condition E1 follows as a consequence of E3 and E4, as discussed in Section 2.

We now turn to the definition of a sequence of replica economies of $\mathcal{E}$. As the reader will see in Example 3, there are alternative ways to construct a sequence of replica economies of $\mathcal{E}$, potentially leading to different associated sequences $(W_r(\mathcal{E}))_{r\in\mathbb{N}}$.

For each $r \in \mathbb{N}$, define the following correspondence $Y_r : 2^{\mathcal{I}_r} \rightrightarrows \mathbb{R}^2$:

$$Y_r(S) = \min\{\kappa^A(S), \kappa^B(S)\}Y(\{A, B\}) + |\kappa^B(S) - \kappa^A(S)|Y(\{A\}),$$

if $S \neq \varnothing$ and $Y_r(S) = \{\mathbf{0}\}$ otherwise.

Claim 1 in Appendix A establishes that Y_r is superadditive for all $r \in \mathbb{N}$. All remaining assumptions required for $\mathcal{E}_r = \left(\mathcal{I}_r, (\mathbf{e}^i)_{(i,q)\in\mathcal{I}_r}, (\succsim^i)_{(i,q)\in\mathcal{I}_r}, Y_r\right)$ to be a sequence of replica economies of $\mathcal{E}$ are straightforward to verify. With all conditions met as defined, we now show that the associated sequence $(W_r(\mathcal{E}))_{r\in\mathbb{N}}$ is constant by proving that $W_r(\mathcal{E}) = W_{r+1}(\mathcal{E})$ for all $r \geq 1$.

First, starting from a Walras equilibrium $({}^r\mathbf{x}, {}^r\mathbf{t}, \mathbf{p}, r\mathbf{y})$ of $\mathcal{E}_r$, we demonstrate that $({}^{r+1}\mathbf{x}, {}^{r+1}\mathbf{t}, \mathbf{p}, (r+1)\,\mathbf{y})$ is a Walrasian equilibrium of $\mathcal{E}_{r+1}$. Conditions E1, E3 and E4 are straightforward to verify. To check E2, we consider any subset $S \subseteq \mathcal{I}_{r+1}$ and show that $\sum_{(i,q)} t^i \geq \Pi_{\mathcal{E}_{r+1},p}(S)$. If $S \subseteq \mathcal{I}_r$, the condition is satisfied. If S is not contained in $\mathcal{I}_r$, we can represent S as the union of two disjoint coalitions, S_1 and S_2, such that $S_1 \subseteq \mathcal{I}_r$ and $S_2 \subseteq \mathcal{I}_{r+1} \setminus \mathcal{I}_r$. Note that S_2 may contain only one individual of type A, only one of type B, or one of each type. In any of these cases, the specific definition Y_{r+1} allows us to write the production possibility set of S as the sum of the production possibility sets of S_1 and S_2. That is,

$$Y_{r+1}(S) = Y_r(S_1) + Y_r(S_2).$$

This implies the following

$$\Pi_{\mathcal{E}_{r+1},p}(S) = \sup_{\boldsymbol{v}\in Y_r(S_1)} p\cdot\boldsymbol{v} + \sup_{\boldsymbol{v}\in Y_r(S_2)} p\cdot\boldsymbol{v} \leq \sum_{(i,q)\in S_1} t^i + \sum_{(i,q)\in S_2} t^i = \sum_{(i,q)\in S} t^i,$$

where the inequality comes from the fact that ${}^r\mathbf{t}$ is a profit distribution of equilibrium in $\mathcal{E}_r$.

Conversely, demonstrating that if $({}^{r+1}\mathbf{x}, {}^{r+1}\mathbf{t}, \mathbf{p}, (r+1)\mathbf{y})$ constitutes a Walrasian equilibrium in $\mathcal{E}_{r+1}$, then $({}^r\mathbf{x}, {}^r\mathbf{t}, \mathbf{p}, r\mathbf{y})$ also represents a Walrasian equilibrium in $\mathcal{E}_r$, is straightforward. It simply requires an individual examination of each of the four conditions.

As a conclusion, since $\mathbf{x} = ((0, 5/2), (0, 3/2)) \in W(\mathcal{E})$, it also holds that $\mathbf{x} \in \bigcap_{r=1}^{\infty} W_r(\mathcal{E})$.

Example 3. Consider the economy $\mathcal{E}$ defined in Example 2. To examine a setting that allows for externalities in production, we now derive a replication structure without inter-replica additivity. For each $r \in \mathbb{N}$, define the following correspondence $Y_r : 2^{\mathcal{I}_r} \rightrightarrows \mathbb{R}^2$:

$$Y_r(S) = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0,\, x_2 \leq -x_1 \text{ and } x_2 \leq f(\kappa(S))\right\},$$

if $S \neq \varnothing$ and $Y_r(S) = \{\mathbf{0}\}$ otherwise, where $f : \mathbb{Z}_+^2 \to \mathbb{R}$ is given by $f(a,b) = \left(\sqrt{a} + \sqrt{b}\right)^2$.

It is straightforward to verify that the sequence $\mathcal{E}_r = \left(\mathcal{I}_r, (\mathbf{e}^i)_{(i,q)\in\mathcal{I}_r}, (\succsim^i)_{(i,q)\in\mathcal{I}_r}, Y_r\right)$ satisfies A4 and A5. To confirm that $(\mathcal{E}_r)_{r\in\mathbb{N}}$ is indeed a sequence of replica economies of $\mathcal{E}$, it suffices to verify that each Y_r is superadditive and homogeneous.

Let $r \in \mathbb{N}$. Note that, given two non-empty coalitions S and S', the Minkowski sum $Y_r(S) + Y_r(S')$ is the set

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0,\, x_2 \leq -x_1 \text{ and } x_2 \leq f(\kappa(S)) + f(\kappa(S'))\right\},$$

and, given $\alpha > 0$, $\alpha Y_r(S)$ is the set

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0,\, x_2 \leq -x_1 \text{ and } x_2 \leq \alpha f(\kappa(S))\right\}.$$

That is, the production sets defined by Y_r not only share the same shape, but also have heights (given by the function f) that behave linearly under addition and scalar multiplication. Thus, the homogeneity and superadditivity of Y_r are a consequence of the corresponding properties of the function f: homogeneity is immediate, whereas superadditivity is established in Claim 2 in Appendix A.

From Example 2, we know that $\mathbf{x} = ((0, 5/2), (0, 3/2))$ is a Walrasian equilibrium allocation of $\mathcal{E}$. In fact, the replication structure considered there ensures that $\mathbf{x}$ remains a Walrasian equilibrium allocation in every replica of $\mathcal{E}$. However, under the replication structure considered here, ${}^2\mathbf{x}$ does not belong to $W(\mathcal{E}_2)$—and thus belongs only to $W(\mathcal{E})$. To see this, note that ${}^2\mathbf{x}$ is blocked by any coalition with profile $(1, 2)$, since such a coalition can produce $\left(1 + \sqrt{2}\right)^2$ units of good 2. With this amount, the coalition can allocate the bundle $\left(0, \frac{5}{2}\right)$ to its type-A member and $\left(0, \frac{(1+\sqrt{2})^2 - 5/2}{2}\right)$ to each of its two type-B members, thereby strictly improving their utilities. By the First Welfare Theorem, this implies that ${}^2\mathbf{x}$ is not a Walrasian equilibrium allocation in $\mathcal{E}_2$.

Claim 1. For all $r \in \mathbb{N}$, the Y_r correspondence defined in Example 2 is superadditive.

Proof. Let $S, S' \in 2^{\mathcal{I}_r}$. We aim to prove that

$$Y_r(S) + Y_r(S') \subseteq Y_r(S + S').$$

The statement is straightforward when either S or S' are empty coalitions. By considering cases based on whether $\kappa^A(S) = \kappa^B(S)$, $\kappa^A(S) < \kappa^B(S)$, or $\kappa^A(S) > \kappa^B(S)$ — and similarly for $\kappa^A(S')$ and $\kappa^B(S')$ — one can also verify that the inclusion above is actually an equality, except when the relations are opposed, that is, when ($\kappa^A(S) < \kappa^B(S)$ and $\kappa^A(S') > \kappa^B(S')$) or ($\kappa^A(S) > \kappa^B(S)$ and $\kappa^A(S') < \kappa^B(S')$). Let us therefore focus on these cases.

Without loss of generality, suppose that $\kappa^A(S) < \kappa^B(S)$ and $\kappa^A(S') > \kappa^B(S')$. Consider first the case where $\kappa^A(S) + \kappa^A(S') = \kappa^B(S) + \kappa^B(S')$. By the definition of Y_r,

$$Y_r(S + S') = (\kappa^A(S) + \kappa^A(S'))Y(\{A, B\}).$$

By substituting $\kappa^A(S) + \kappa^A(S')$ by $(\kappa^A(S) + \kappa^B(S')) + (\kappa^A(S') - \kappa^B(S'))$ one obtains

$$(\kappa^A(S) + \kappa^B(S'))Y(\{A, B\}) + (\kappa^A(S') - \kappa^B(S'))Y(\{A, B\})$$

Since $\kappa^B(S) - \kappa^A(S) = \kappa^A(S') - \kappa^B(S')$, this expression equals

$$(\kappa^A(S) + \kappa^B(S'))Y(\{A, B\}) + (\kappa^B(S) - \kappa^A(S))Y(\{A, B\}).$$

Observe from the definitions of $Y(\{A, B\})$ and $Y(\{A\})$ that $2Y(\{A\}) \subseteq Y(\{A, B\})$. Thus, the last expression contains

$$(\kappa^A(S) + \kappa^B(S'))Y(\{A, B\}) + 2(\kappa^B(S) - \kappa^A(S))Y(\{A\}),$$

precisely $Y_r(S) + Y_r(S')$.

On the other hand, assume, again without loss of generality, that $\kappa^A(S) + \kappa^A(S') < \kappa^B(S) + \kappa^B(S')$. By definition, $Y_r(S + S')$ is equal to

$$(\kappa^A(S) + \kappa^A(S'))Y(\{A, B\}) + \left(\kappa^B(S) + \kappa^B(S') - \kappa^A(S) - \kappa^A(S')\right) Y\{A\}.$$

By rewriting $\kappa^A(S) + \kappa^A(S')$ as $(\kappa^A(S) + \kappa^B(S')) + (\kappa^A(S') - \kappa^B(S'))$, and using the fact that $2Y(\{A\}) \subseteq Y(\{A, B\})$, we find that the resulting expression contains:

$$\begin{aligned}\left(\kappa^A(S) + \kappa^B(S')\right) Y(\{A, B\}) + 2\left(\kappa^A(S') - \kappa^B(S')\right) Y(\{A\}) \\ + \left(\kappa^B(S) + \kappa^B(S') - \kappa^A(S) - \kappa^A(S')\right) Y\{A\},\end{aligned}$$

which, in turn, is equal to $Y_r(S) + Y_r(S')$. □

Claim 2. The function $f : \mathbb{Z}_+^2 \to \mathbb{R}$ given by $f(a, b) = \left(\sqrt{a} + \sqrt{b}\right)^2$ is superadditive.

Proof. Let $(a, b), (a', b') \in \mathbb{Z}_+^2$ be given. In order to check that $f(a + a', b + b') \geq f(a, b) + f(a', b')$, note that there would be nothing to prove if $(a, b) = (0, 0)$ or $(a', b') = (0, 0)$. We may thus assume that none of these ordered pairs is the null vector, so that $\lambda := f(a', b') / (f(a, b) + f(a', b'))$ must belong to the $(0, 1)$ interval.

Concavity of $\sqrt{\cdot}$ now gives

$$\sqrt{a+a'} = \sqrt{(1-\lambda)\frac{a}{1-\lambda}+\lambda\frac{a'}{\lambda}} \geq (1-\lambda)\sqrt{\frac{a}{1-\lambda}}+\lambda\sqrt{\frac{a'}{\lambda}} = \sqrt{1-\lambda}\sqrt{a}+\sqrt{\lambda}\sqrt{a'}$$

and, similarly,

$$\sqrt{b+b'} \geq \sqrt{1-\lambda}\sqrt{b}+\sqrt{\lambda}\sqrt{b'}.$$

Therefore,

$$\begin{aligned}
& \sqrt{f\left(a+a',b+b'\right)} = \sqrt{a+a'}+\sqrt{b+b'} \geq \sqrt{1-\lambda}\left(\sqrt{a}+\sqrt{b}\right)+\sqrt{\lambda}\left(\sqrt{a'}+\sqrt{b'}\right) \\
= \; & \sqrt{(1-\lambda)\, f\left(a,b\right)}+\sqrt{\lambda f\left(a',b'\right)} \\
= \; & f\left(a,b\right)\sqrt{\frac{1}{f\left(a,b\right)+f\left(a',b'\right)}}+f\left(a',b'\right)\sqrt{\frac{1}{f\left(a,b\right)+f\left(a',b'\right)}} \\
= \; & \sqrt{f\left(a,b\right)+f\left(a',b'\right)},
\end{aligned}$$

whence it is a matter of simply squaring each side to obtain the sought after inequality. □

B Appendix

Claim 3. The correspondence $Y_{\mathbb{Q}}$, introduced in Section 2, is well-defined, inaction-allowing, convex-valued, superadditive, and homogeneous over $\mathbb{Q}_+$.

Proof. Let $\boldsymbol{\chi} \in \mathbb{Q}_+^I$ and let N be a positive integer such that $N\boldsymbol{\chi} \in \mathbb{Z}_+^I$. The unambiguity in $Y_{\mathbb{Q}}$'s definition results from the observation that N is necessarily of the form nN', where n is a natural number and N' is the least common multiple of the denominators of the coordinates of $\boldsymbol{\chi}$ in their irreducible form, so that the homogeneity of $Y_{\mathbb{Z}}$ implies

$$\frac{1}{nN'}Y_{\mathbb{Z}}(nN'\boldsymbol{\chi}) = \frac{1}{nN'}nY_{\mathbb{Z}}(N'\boldsymbol{\chi}) = \frac{1}{N'}Y_{\mathbb{Z}}(N'\boldsymbol{\chi}).$$

The inaction-allowing and convex-valued properties of $Y_{\mathbb{Q}}$ follow directly from the corresponding properties of $Y_{\mathbb{Z}}$. We proceed to prove its homogeneity over $\mathbb{Q}_+$. Let $\alpha \in \mathbb{Q}_+$. If $\alpha = 0$, since $Y_{\mathbb{Q}}\left(\boldsymbol{\chi}\right)$ is not empty (it contains $\mathbf{0}$), both $\alpha Y_{\mathbb{Q}}\left(\boldsymbol{\chi}\right)$ and $Y_{\mathbb{Q}}\left(\alpha\boldsymbol{\chi}\right)$ equal $\{\mathbf{0}\}$. If $\alpha \neq 0$, choose $M \in \mathbb{N}$ such that $M\alpha \in \mathbb{Z}_+$ and $M\alpha\boldsymbol{\chi} \in \mathbb{Z}_+^I$ at the same time. Then:

$$\alpha Y_{\mathbb{Q}}(\boldsymbol{\chi}) = (\alpha/M\alpha)Y_{\mathbb{Z}}(M\alpha\boldsymbol{\chi}) = (1/M)Y_{\mathbb{Z}}(M\alpha\boldsymbol{\chi}) = Y_{\mathbb{Q}}(\alpha\boldsymbol{\chi}).$$

Finally, we verify superadditivity. Let $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2 \in \mathbb{Q}_+^I$ and take $\bar{N} \in \mathbb{N}$ such that $\bar{N}\boldsymbol{\chi}_1$ and $\bar{N}\boldsymbol{\chi}_2$ are both vectors in $\mathbb{Z}_+^I$. By applying the definition of $Y_{\mathbb{Q}}$ to each term of the sum $Y_{\mathbb{Q}}(\boldsymbol{\chi}_1)+Y_{\mathbb{Q}}(\boldsymbol{\chi}_2)$, we see that it coincides with $\left(1/\bar{N}\right) Y_{\mathbb{Z}}\left(\bar{N}\boldsymbol{\chi}_1\right)+\left(1/\bar{N}\right) Y_{\mathbb{Z}}\left(\bar{N}\boldsymbol{\chi}_2\right)$, which, in turn, equals $\left(1/\bar{N}\right)\left(Y_{\mathbb{Z}}\left(\bar{N}\boldsymbol{\chi}_1\right)+Y_{\mathbb{Z}}\left(\bar{N}\boldsymbol{\chi}_2\right)\right)$. By superadditivity of $Y_{\mathbb{Z}}$, the last expression is contained in $\left(1/\bar{N}\right) Y_{\mathbb{Z}}\left(\bar{N}\left(\boldsymbol{\chi}_1+\boldsymbol{\chi}_2\right)\right)$, precisely the definition of $Y_{\mathbb{Q}}(\boldsymbol{\chi}_1+\boldsymbol{\chi}_2)$.

□

Claim 4. The $Y_{\mathbb{R}}$ correspondence defined in the proof of theorem 2 is homogeneous, that is, for all $\mathbf{s} \in \mathbb{R}_+^I$ and $\alpha \geq 0$, $\alpha Y_{\mathbb{R}}(\mathbf{s}) = Y_{\mathbb{R}}(\alpha\mathbf{s})$.

Proof. We begin by proving the inclusion $\alpha Y_{\mathbb{R}}(\mathbf{s}) \subseteq Y_{\mathbb{R}}(\alpha\mathbf{s})$. Let $\mathbf{z} \in \alpha Y_{\mathbb{R}}(\mathbf{s})$. Then, there exists $\mathbf{y} \in Y_{\mathbb{R}}(\mathbf{s})$ for which $\mathbf{z} = \alpha\mathbf{y}$. Since $\mathbf{y} \in Y_{\mathbb{R}}(\mathbf{s})$, there is a sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ taking values in $\bigcup_{\boldsymbol{\chi}\in\mathbb{Q}_+^I,\,\boldsymbol{\chi}\leq\mathbf{s}} Y_{\mathbb{Q}}(\boldsymbol{\chi})$ such that $\lim \mathbf{y}_n = \mathbf{y}$. For each $n \in \mathbb{N}$, let $\boldsymbol{\chi}_n \leq \mathbf{s}$ be a rational profile such that $\mathbf{y}_n \in Y_{\mathbb{Q}}(\boldsymbol{\chi}_n)$. Define $(\alpha_n)_{n\in\mathbb{N}}$ as a rational sequence such that $\alpha_n \leq \alpha$ for all $n \in \mathbb{N}$ and $\lim \alpha_n = \alpha$ (e.g., the sequence of decimal expansions of α).

The sequence $(\alpha_n\mathbf{y}_n)_{n\in\mathbb{N}}$ has limit $\alpha\mathbf{y}$ and, by homogeneity of $Y_{\mathbb{Q}}$, $\alpha_n\mathbf{y}_n \in Y_{\mathbb{Q}}(\alpha_n\boldsymbol{\chi}_n)$ for all $n \in \mathbb{N}$. Since $\alpha_n\boldsymbol{\chi}_n \leq \alpha\mathbf{s}$ for all $n \in \mathbb{N}$, it follows that

$$\alpha\mathbf{y} \in \overline{\bigcup_{\boldsymbol{\chi}\in\mathbb{Q}_+^I,\,\boldsymbol{\chi}\leq\alpha\mathbf{s}} Y_{\mathbb{Q}}(\boldsymbol{\chi})} = Y_{\mathbb{R}}(\alpha\mathbf{s}).$$

Conversely, since α is arbitrary, we can replace α with $(1/\alpha)$ and $Y_{\mathbb{R}}(\mathbf{s})$ with $Y_{\mathbb{R}}(\alpha\mathbf{s})$ in the previously established direction to deduce that $(1/\alpha)Y_{\mathbb{R}}(\alpha\mathbf{s}) \subseteq Y_{\mathbb{R}}(\mathbf{s})$, which in turn implies $Y_{\mathbb{R}}(\alpha\mathbf{s}) \subseteq \alpha Y_{\mathbb{R}}(\mathbf{s})$. □

Claim 5. The $Y_{\mathbb{R}}$ correspondence defined in the proof of theorem 2 is superadditive, that is, $Y_{\mathbb{R}}(\mathbf{s}^1) + Y_{\mathbb{R}}(\mathbf{s}^2) \subseteq Y_{\mathbb{R}}(\mathbf{s}^1 + \mathbf{s}^2)$ for all $\mathbf{s}^1, \mathbf{s}^2 \in \mathbb{R}_+^I$.

Proof. Let $\mathbf{y}_1 \in Y_{\mathbb{R}}(\mathbf{s}^1)$ and $\mathbf{y}_2 \in Y_{\mathbb{R}}(\mathbf{s}^2)$.

Let $(\mathbf{y}_{1,n})_n$ be a sequence converging to $\mathbf{y}_1$ such that, for all $n \in \mathbb{N}$, $\mathbf{y}_{1,n} \in Y_{\mathbb{Q}}\left(\boldsymbol{\chi}_{1,n}\right)$ for some rational profile $\boldsymbol{\chi}_{1,n} \leq \mathbf{s}^1$.[30]

[30] Note that $\mathbf{y} \in Y_{\mathbb{R}}(\mathbf{s})$ means that $\mathbf{y}$ is the limit point of a sequence taking values in $\bigcup_{\boldsymbol{\chi}\in\mathbb{Q}_+^I,\,\boldsymbol{\chi}\leq\mathbf{s}} Y_{\mathbb{Q}}(\boldsymbol{\chi})$, that is, there exists a sequence of rational profiles $(\boldsymbol{\chi}_n)_{n\in\mathbb{N}}$ coordinatewise bounded above by $\mathbf{s}$ and a sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ such that $\mathbf{y}_n \in Y_{\mathbb{Q}}(\boldsymbol{\chi}_n)$, for all $n \in \mathbb{N}$, and $\mathbf{y}_n \to \mathbf{y}$.

Similarly, let $(\mathbf{y}_{2,n})_n$ be a sequence converging to $\mathbf{y}_2$ such that, for all $n \in \mathbb{N}$, $\mathbf{y}_{2,n} \in Y_{\mathbb{Q}}\left(\boldsymbol{\chi}_{2,n}\right)$ for some rational profile $\boldsymbol{\chi}_{2,n} \leq \mathbf{s}^2$.

Note that $\mathbf{y}_{1,n} + \mathbf{y}_{2,n} \in Y_{\mathbb{Q}}\left(\boldsymbol{\chi}_{1,n} + \boldsymbol{\chi}_{2,n}\right)$ for all $n \in \mathbb{N}$, because the supperadditivity of $Y_{\mathbb{Q}}$ is implied by the superadditivity of $Y_{\mathbb{Z}}$. Since $\mathbf{y}_1 + \mathbf{y}_2 = \lim\left(\mathbf{y}_{1,n} + \mathbf{y}_{2,n}\right)$,

$$\mathbf{y}_1 + \mathbf{y}_2 \in \overline{\bigcup_{\boldsymbol{\chi} \leq \mathbf{s}^1 + \mathbf{s}^2,\, \boldsymbol{\chi} \in \mathbb{Q}_+^I} Y_{\mathbb{Q}}(\boldsymbol{\chi})} = Y_{\mathbb{R}}\left(\mathbf{s}^1 + \mathbf{s}^2\right).$$

□

Claim 6. The set Z, as defined in the proof of theorem 2, is a convex cone.

Proof. First, note that Z contains the origin, since $Y_{\mathbb{R}}(\mathbf{0}) = \{\mathbf{0}\}$. We proceed to establish that Z is convex. Let $(\mathbf{y}_1, -\mathbf{s}_1) \in Z$, $(\mathbf{y}_2, -\mathbf{s}_2) \in Z$ and $\alpha \in (0,1)$. Note that

$$\alpha(\mathbf{y}_1, -\mathbf{s}_1) + (1-\alpha)(\mathbf{y}_2, -\mathbf{s}_2) = (\alpha\mathbf{y}_1 + (1-\alpha)\mathbf{y}_2, -(\alpha\mathbf{s}_1 + (1-\alpha)\mathbf{s}_2)).$$

By Claim 4,

$$\alpha\mathbf{y}_1 + (1-\alpha)\mathbf{y}_2 \in \alpha Y_{\mathbb{R}}(\mathbf{s}_1) + (1-\alpha) Y_{\mathbb{R}}(\mathbf{s}_2) \subseteq Y_{\mathbb{R}}(\alpha\mathbf{s}_1) + Y_{\mathbb{R}}((1-\alpha)\mathbf{s}_2).$$

Now, by Claim 5,

$$Y_{\mathbb{R}}(\alpha\mathbf{s}_1) + Y_{\mathbb{R}}((1-\alpha)\mathbf{s}_2) \subseteq Y_{\mathbb{R}}(\alpha\mathbf{s}_1 + (1-\alpha)\mathbf{s}_2).$$

Combining the above, we deduce that, $\alpha(\mathbf{y}_1, -\mathbf{s}_1) + (1-\alpha)(\mathbf{y}_2, -\mathbf{s}_2) \in Z$. Thus, Z is convex.

To see that Z is a cone, let $c \geq 0$ and $(\mathbf{y}, -\mathbf{s}) \in Z$. By claim 2, we know that $c\mathbf{y} \in Y_{\mathbb{R}}(c\mathbf{s})$. Therefore, $c(\mathbf{y}, -\mathbf{s}) \in Z$. □

Claim 7. In the context of the proof of theorem 2, if $(\mathbf{x}^i)_{i\in\mathcal{I}} \in \mathbb{R}_+^{LI}$ is an Edgeworth equilibrium of $\mathcal{E}$, then ${}^r(\mathbf{x}^i, \mathbf{0})_{i\in\mathcal{I}} \in C(\mathcal{E}_r')$.

Proof. Let $r \geq 1$. First, note that $(\mathbf{x}^i, \mathbf{0})_{(i,q)\in\mathcal{I}_r}$ is necessarily feasible in $\mathcal{E}_r'$. In fact, since $Y_{\mathbb{R}}$ and $Y_{\mathbb{Z}}$ coincide over integer-valued vectors, $\sum_{i\in\mathcal{I}}(\mathbf{x}^i - \mathbf{e}^i) \in Y_{\mathbb{Z}}\left(\sum_{i\in\mathcal{I}}\mathbf{u}^i\right) = Y_{\mathbb{R}}\left(\sum_{i\in\mathcal{I}}\mathbf{u}^i\right)$, so that $\left(\sum_{i\in\mathcal{I}}(\mathbf{x}^i - \mathbf{e}^i), -\sum_{i\in\mathcal{I}}\mathbf{u}^i\right) \in Z$, that is, $\sum_{i\in\mathcal{I}}(\mathbf{x}^i, \mathbf{0}) - \sum_{i\in\mathcal{I}}\mathbf{f}^i \in Z$. By Claim 6, $r\sum_{i\in\mathcal{I}}(\mathbf{x}^i, \mathbf{0}) - r\sum_{i\in\mathcal{I}}\mathbf{f}^i \in Z$.

Assume that there is a coalition S in $\mathcal{E}'_r$ that blocks $(\mathbf{x}^i, \mathbf{0})_{(i,q)\in\mathcal{I}_r}$. Thus, there is an allocation $\left(\mathbf{x}'^{(i,q)}, \mathbf{v}'^{(i,q)}\right)$ such that $\left(\mathbf{x}'^{(i,q)}, \mathbf{v}'^{(i,q)}\right) \rhd^i (\mathbf{x}^i, \mathbf{0})$ for all $(i,q) \in S$ and $\sum_{(i,q)\in S}\left(\left(\mathbf{x}'^{(i,q)}, \mathbf{v}'^{(i,q)}\right) - (\mathbf{e}^i, \mathbf{u}^i)\right) \in Z$. It follows that

$$\sum_{(i,q)\in S}\left(\mathbf{x}'^{(i,q)} - \mathbf{e}^i\right) \in Y_{\mathbb{R}}\left(\sum_{(i,q)\in S}\left(\mathbf{u}^i - \mathbf{v}'^{(i,q)}\right)\right) \subseteq Y_{\mathbb{R}}\left(\sum_{(i,q)\in S}\mathbf{u}^i\right),$$

where the last inclusion comes from the fact that $Y_{\mathbb{R}}$ is monotone.

Therefore, S blocks $(\mathbf{x}^i)_{i\in\mathcal{I}_r}$ in $\mathcal{E}_r$, a contradiction. $\square$

Claim 8. The price vector $\mathbf{p}$, as defined in the proof of theorem 2, is strictly positive, that is, $\mathbf{p} \gg \mathbf{0}$.

Proof. First, we posit that $\mathbf{p}$ must be nonnegative, i.e., $\mathbf{p} \geq 0$. To elucidate, assume, for the sake of contradiction, that $p_j < 0$ for some $j \in \{1, 2, ..., L\}$. If we consider the bundle $\mathbf{x}^i + \mathbf{u}^j$, then $\mathbf{p} \cdot (\mathbf{x}^i + \mathbf{u}^j) < \mathbf{p} \cdot \mathbf{x}^i$ and, by strong monotonicity, $\mathbf{x}^i + \mathbf{u}^j \succ^i \mathbf{x}^i$, a contradiction. Now note that at least one entry of $\mathbf{p}$ must be positive since, otherwise, for each $i \in \mathcal{I}$, $t^i = \mathbf{p} \cdot (\mathbf{x}^i - \mathbf{e}^i)$, and $(\mathbf{p}, \mathbf{t}) = \mathbf{0}$, a contradiction.

We now establish that not just some, but all, coordinates of $\mathbf{p}$ are strictly positive. We finally make use of technical assumption TA2 involving $\widetilde{\mathbf{y}}$. Given the validity of (E.2), it is known that $\sum_{i\in\mathcal{I}} t^i \geq \mathbf{p} \cdot \widetilde{\mathbf{y}}$. Moreover, since $\sum_{i\in\mathcal{I}} t^i = \sum_{i\in\mathcal{I}} \mathbf{p} \cdot (\mathbf{x}^i - \mathbf{e}^i)$,

$$\mathbf{p} \cdot \sum_{i\in\mathcal{I}} \mathbf{x}^i \geq \mathbf{p} \cdot \left(\sum_{i\in\mathcal{I}} \mathbf{e}^i + \widetilde{\mathbf{y}}\right) > 0,$$

because $\mathbf{p} \ngeq \mathbf{0}$ and $\sum_{i\in\mathcal{I}} \mathbf{e}^i + \widetilde{\mathbf{y}} \gg \mathbf{0}$. Therefore, there is a coordinate $l \in \{1, ..., L\}$ such that p_l and $\sum_{i\in\mathcal{I}} x^i_l$ are both strictly positive, implying $x^{\bar{i}}_l > 0$ for some individual $\bar{i}$.

Assume toward another contradiction that $p_k = 0$ for some coordinate k. Define

$$\overline{\mathbf{x}}^{\bar{i}} = \mathbf{x}^{\bar{i}} + \mathbf{u}_k,$$

so that strong monotonicity gives $\overline{\mathbf{x}}^{\bar{i}} \succ^{\bar{i}} \mathbf{x}^{\bar{i}}$. Now, let

$$\widehat{\mathbf{x}}^{\bar{i}} = \overline{\mathbf{x}}^{\bar{i}} - \delta \mathbf{u}_l,$$

where δ is sufficiently small to ensure not only that $\widehat{\mathbf{x}}^{\bar{i}}$ is still a consumption bundle, but also that $\widehat{\mathbf{x}}^{\bar{i}} \succ^{\bar{i}} \mathbf{x}^{\bar{i}}$, which is possible due to lower semicontinuity of preferences.

Therefore, $\left(\widehat{\mathbf{x}}^{\bar{i}}, \mathbf{0}\right) - \mathbf{f}^{\bar{i}} \in \Gamma^{\bar{i}} \subseteq \Gamma$. From (2), $(\mathbf{p}, \mathbf{t}) \cdot \left(\left(\widehat{\mathbf{x}}^{\bar{i}}, \mathbf{0}\right) - \mathbf{f}^{\bar{i}}\right) \geq 0$. Yet, observe that

$$\begin{aligned}
(\mathbf{p}, \mathbf{t}) \cdot \left(\left(\widehat{\mathbf{x}}^{\bar{i}}, \mathbf{0}\right) - \mathbf{f}^{i}\right) &= \mathbf{p} \cdot \widehat{\mathbf{x}}^{\bar{i}} - \mathbf{p} \cdot \mathbf{e}^{\bar{i}} - t^{\bar{i}} \\
&= \mathbf{p} \cdot \overline{\mathbf{x}}^{\bar{i}} - \delta p_l - \mathbf{p} \cdot \mathbf{e}^{\bar{i}} - t^{\bar{i}} \\
&= \mathbf{p} \cdot \mathbf{x}^{\bar{i}} + p_k - \delta p_l - \mathbf{p} \cdot \mathbf{e}^{\bar{i}} - t^{\bar{i}} \\
&= p_k - \delta p_l < 0,
\end{aligned}$$

a contradiction that completes the proof. □

Claim 9. The $Y_{\mathbb{R}}$ correspondence defined in the proof of theorem 4 is homogeneous, that is, for all $\mathbf{s} \in \mathbb{R}^I_+$ and $\alpha \geq 0$, $\alpha Y_{\mathbb{R}}(\mathbf{s}) = Y_{\mathbb{R}}(\alpha \mathbf{s})$.

Proof. Let $\mathbf{s} \in \mathbb{R}^I_+$ and $\alpha \geq 0$. The $\alpha = 0$ and the $\mathbf{s} = \mathbf{0}$ cases are straightforward. If $\alpha > 0$ and $\mathbf{s} \neq \mathbf{0}$, note that

$$Y_{\mathbb{R}}(\alpha \mathbf{s}) = \|\alpha \mathbf{s}\|_1 Y_f \left(\frac{1}{\|\alpha \mathbf{s}\|_1} \alpha \mathbf{s}\right) = \alpha \|\mathbf{s}\|_1 Y_f \left(\frac{1}{\|\mathbf{s}\|_1} \mathbf{s}\right) = \alpha Y_{\mathbb{R}}(\mathbf{s}).$$

□

Claim 10. The $Y_{\mathbb{R}}$ correspondence defined in the proof of theorem 4 is an extension of Y_f, that is, $Y_{\mathbb{R}}|_{[0,1]^I} = Y_f$.

Proof. Let $\mathbf{s} \in [0,1]^I$. Without loss of generality, $\mathbf{s} \neq \mathbf{0}$. By definition of $Y_{\mathbb{R}}$,

$$Y_{\mathbb{R}}(\mathbf{s}) = \|\mathbf{s}\|_1 Y_f \left(\frac{1}{\|\mathbf{s}\|_1} \mathbf{s}\right).$$

If $\|\mathbf{s}\|_1 \geq 1$, the homogeneity of Y_f implies

$$Y_f \left(\frac{1}{\|\mathbf{s}\|_1} \mathbf{s}\right) = \frac{1}{\|\mathbf{s}\|_1} Y_f (\mathbf{s}),$$

thus concluding the proof in this case. If $\|\mathbf{s}\|_1 < 1$, the homogeneity of Y_f yields, since $\frac{1}{\|\mathbf{s}\|_1}\mathbf{s}$ is still a fuzzy coalition,

$$\|\mathbf{s}\|_1 Y_f \left(\frac{1}{\|\mathbf{s}\|_1} \mathbf{s}\right) = Y_f (\mathbf{s}).$$

□

Claim 11. The $Y_{\mathbb{R}}$ correspondence defined in the proof of theorem 4 is superadditive, that is, $Y_{\mathbb{R}}(\mathbf{s}_1) + Y_{\mathbb{R}}(\mathbf{s}_2) \subseteq Y_{\mathbb{R}}(\mathbf{s}_1 + \mathbf{s}_2)$ for all $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^I_+$.

Proof. Let $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^I_+$. If $\mathbf{s}_1 + \mathbf{s}_2 = \mathbf{0}$, then $\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{0}$, and we are done. We thus assume $\mathbf{s}_1 + \mathbf{s}_2 \neq \mathbf{0}$. By definition of $Y_{\mathbb{R}}$,

$$Y_{\mathbb{R}}(\mathbf{s}_1 + \mathbf{s}_2) = \|\mathbf{s}_1 + \mathbf{s}_2\|_1 Y_f \left(\frac{1}{\|\mathbf{s}_1 + \mathbf{s}_2\|_1} (\mathbf{s}_1 + \mathbf{s}_2) \right)$$

which, by superadditivity of Y_f, contains

$$\|\mathbf{s}_1 + \mathbf{s}_2\|_1 \left(Y_f \left(\frac{1}{\|\mathbf{s}_1 + \mathbf{s}_2\|_1} \mathbf{s}_1 \right) + Y_f \left(\frac{1}{\|\mathbf{s}_1 + \mathbf{s}_2\|_1} \mathbf{s}_2 \right) \right).$$

By Claims 9 and 10, the last expression equals

$$\|\mathbf{s}_1 + \mathbf{s}_2\|_1 \left(\frac{1}{\|\mathbf{s}_1 + \mathbf{s}_2\|_1} Y_{\mathbb{R}} (\mathbf{s}_1) + \frac{1}{\|\mathbf{s}_1 + \mathbf{s}_2\|_1} Y_{\mathbb{R}} (\mathbf{s}_2) \right),$$

which, in turn, is equal to $Y_{\mathbb{R}} (\mathbf{s}_1) + Y_{\mathbb{R}} (\mathbf{s}_2)$. □

Claim 12. The set Z, as defined in the proof of theorem 4, is a convex cone.

Proof. First, note that Z contains the origin, since $Y_{\mathbb{R}} (\mathbf{0}) = \{\mathbf{0}\}$. We proceed to establish that Z is convex. Let $(\mathbf{y}_1, -\mathbf{s}_1) \in Z$, $(\mathbf{y}_2, -\mathbf{s}_2) \in Z$ and $\alpha \in (0, 1)$. Note that

$$\alpha (\mathbf{y}_1, -\mathbf{s}_1) + (1 - \alpha) (\mathbf{y}_2, -\mathbf{s}_2) = (\alpha \mathbf{y}_1 + (1 - \alpha) \mathbf{y}_2, -(\alpha \mathbf{s}_1 + (1 - \alpha) \mathbf{s}_2)).$$

By Claim 9,

$$\alpha \mathbf{y}_1 + (1 - \alpha) \mathbf{y}_2 \in \alpha Y_{\mathbb{R}} \left(\mathbf{s}^1\right) + (1 - \alpha) Y_{\mathbb{R}} \left(\mathbf{s}^2\right) = Y_{\mathbb{R}} \left(\alpha \mathbf{s}^1\right) + Y_{\mathbb{R}} \left((1 - \alpha) \mathbf{s}^2\right).$$

By Claim 11,

$$Y_{\mathbb{R}} \left(\alpha \mathbf{s}^1\right) + Y_{\mathbb{R}} \left((1 - \alpha) \mathbf{s}^2\right) \subseteq Y_{\mathbb{R}} \left(\alpha \mathbf{s}^1 + (1 - \alpha) \mathbf{s}^2\right).$$

Combining the above, we deduce that $\alpha (\mathbf{y}_1, -\mathbf{s}_1) + (1 - \alpha) (\mathbf{y}_2, -\mathbf{s}_2) \in Z$. Thus, Z is convex.

To see that Z is a cone, let $c \geq 0$ and $(\mathbf{y}, -\mathbf{s}) \in Z$. By Claim 9, we know that $c\mathbf{y} \in Y_{\mathbb{R}} (c\mathbf{s})$. Therefore, $c (\mathbf{y}, -\mathbf{s}) \in Z$. □

Claim 13. In the context of the proof of theorem 4, if $(\mathbf{x}^i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}_+$ is a core allocation of $\mathcal{E}^f$, then $(\mathbf{x}^i, \mathbf{0})_{i \in \mathcal{I}} \in \mathbb{R}^{(L+I)I}_+$ is a core allocation of $\mathcal{E}^{f'}$.

Proof. First, note that $(\mathbf{x}^i, \mathbf{0})_{i \in \mathcal{I}}$ is feasible in $\mathcal{E}^{f'}$. In fact, Claim 10 gives $\sum_{i \in \mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i) \in Y_f (\mathbf{1}) = Y_{\mathbb{R}} (\mathbf{1})$, so that $\left(\sum_{i \in \mathcal{I}} (\mathbf{x}^i - \mathbf{e}^i), -\mathbf{1}\right) \in Z$, that is, $\sum_{i \in \mathcal{I}} (\mathbf{x}^i, \mathbf{0}) - \sum_{i \in \mathcal{I}} \mathbf{f}^i \in Z$.

Assume that there is a coalition $\mathbf{s}$ in $\mathcal{E}^{f'}$ that blocks $(\mathbf{x}^i, \mathbf{0})_{i\in\mathcal{I}}$. Thus, there exists an allocation $(\mathbf{x}'^i, \mathbf{v}'^i)_{i\in\mathcal{I}}$ such that $(\mathbf{x}'^i, \mathbf{v}'^i) \triangleright^i (\mathbf{x}^i, \mathbf{0})$ for all $i \in car(\mathbf{s})$ and $\sum_{i\in\mathcal{I}} \left((\mathbf{x}'^i, \mathbf{v}'^i)\right) - (\mathbf{e}^i, \mathbf{u}^i)\big) \in Z$ (so that $\sum_{i\in\mathcal{I}} \mathbf{v}'^i \leq \mathbf{1}$). Again using Claim 10, it follows that

$$\sum_{i\in\mathcal{I}} \left(\mathbf{x}'^i - \mathbf{e}^i\right) \in Y_f \left(\mathbf{1} - \sum_{i\in\mathcal{I}} \mathbf{v}'^i\right) \subseteq Y_f\left(\mathbf{1}\right),$$

where the last inclusion comes from the fact that Y_f is monotone.

Therefore, $\mathbf{s}$ blocks $(\mathbf{x}^i)_{i\in\mathcal{I}}$ in $\mathcal{E}^f$, a contradiction. $\square$